



Useful Solutions for Standard Problems

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Physical constants in SI units

Physical constant	Value in SI units	
Absolute zero temperature	-273.2	°C
Acceleration due to gravity, g	9.807	m/s ²
Avogadro's number, NA	6.022x10 ²³	-
Base of natural logarithms, e	2.718	-
Boltzmann's constant, k	1.381x10 ⁻²³	J/K
Faraday's constant k	9.648x10 ⁴	C/mol
Gas constant,	8.314	J/mol-K
Permeability of vacuum, μ_0	1.257x10 ⁻⁶	H/m
Permittivity of vacuum, ϵ_0	8.854x10 ⁻¹²	F/m
Planck's constant, h	6.626x10 ⁻³⁴	J/s
Velocity of light in vacuum, c	2.998x10 ⁸	m/s
Volume of perfect gas at STP	22.41x10 ⁻³	m ³ /mol

Conversion of units, general

Quantity	Imperial Unit	SI Unit
Angle, θ	1 rad	57.30°
Density, ρ	1 lb/ft ³	16.03 kg/m ³
Diffusion coefficient, D	1cm ³ /s	1.0 x 10 ⁻⁴ m ² /s
Energy, U	See inside back cover	
Force, F	1 kgf	9.807 N
	1 lbf	4.448 N
	1 dyne	1.0 x 10 ⁻⁵ N
Length, l	1 ft	304.8 mm
	1 inch	25.40 mm
	1 Å	0.1 nm
Mass, M	1 tonne	1000 kg
	1 short ton	908 kg
	1 long ton	1107 kg
	1 lb mass	0.454 kg
Power, P	See inside back cover	
Stress, σ	See inside back cover	
Specific heat, C _p	1 cal/gal.°C	4.188 kJ/kg.°C
	Btu/°F	4.187 kg/kg.°C
Stress intensity, K _{1c}	1 ksi $\sqrt{\text{in}}$	1.10 MN/m ^{3/2}
Surface energy γ	1 erg/cm ²	1 mJ/m ²
Temperature, T	1°F	0.556 K
Thermal conductivity λ	1 cal/s.cm.°C	418.8 W/m.°C
	1 Btu/h.ft.°F	1.731 W/m.°C
Volume, V	1 Imperial gall	4.546 x 10 ⁻³ m ³
	1 US gall	3.785 x 10 ⁻³ m ³
Viscosity, η	1 poise	0.1 N.s/m ²
	1 lb ft.s	0.1517 N.s/m ²

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Useful Solutions for Standard Problems

Preface

Modeling is a key part of design. In the early stage, approximate modeling establishes whether the concept will work at all, and identifies the combination of material properties that maximize performance. At the embodiment stage, more accurate modeling brackets values for the forces, the displacements, the velocities, the heat fluxes and the dimensions of the components. And in the final stage, modeling gives precise values for stresses, strains and failure probability in key components; power, speed, efficiency and so forth.

Many components with simple geometries and loads have been modeled already. Many more-complex components can be modeled approximately by idealizing them as one of these. There is no need to reinvent the beam or the column or the pressure vessel; their behavior under all common types of loading has already been analyzed. The important thing is to know that the results exist and where to find them.

This Booklet summarizes the results of modeling a number of standard problems. Their usefulness cannot be overstated. Many problems of conceptual design can be treated, with adequate precision, by patching together the solutions given here; and even the detailed analysis of non-critical components can often be tackled in the same way. Even when this approximate approach is not sufficiently accurate, the insight it gives is valuable.

This Booklet contains 16 double page sections that list, with a short commentary, results for constitutive equations; for the loading of beams, columns and torsion bars; for contact stresses, cracks and other stress concentrations; for pressure vessels, vibrating beams and plates; and for the flow of heat and matter. They are drawn from numerous sources, listed under “Further reading” in Section 17.

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1. Constitutive equations for mechanical response

The behavior of a component when it is loaded depends on the mechanism by which it deforms. A beam loaded in bending may deflect elastically; it may yield plastically; it may deform by creep; and it may fracture in a brittle or in a ductile way. The equation that describes the material response is known as a constitutive equation. Each mechanism is characterized by a different constitutive equation. The constitutive equation contains one or more than one material property: Young's modulus, E , and Poisson's ratio, ν , are the material properties that enter the constitutive equation for linear-elastic deformation; the yield strength, σ_y , is the material property that enters the constitutive equation for plastic flow; creep constants, $\dot{\epsilon}_o$, σ_o , and n enter the equation for creep; the fracture toughness, K_{1c} , enters that for brittle fracture.

The common constitutive equations for mechanical deformation are listed on the facing page. In each case the equation for uniaxial loading by a tensile stress σ is given first; below it is the equation for multiaxial loading by principal stresses σ_1 , σ_2 , and σ_3 , always chosen so that σ_1 is the most tensile and σ_3 the most compressive (or least tensile) stress. They are the basic equations that determine mechanical response.

1. Mechanical constitutive equations

Elastic

Uniaxial

$$\epsilon = \frac{\sigma}{E}$$

General

$$\epsilon_1 = \frac{\sigma_1}{E} - n (\sigma_2 + \sigma_3) \quad \text{etc}$$

Plastic

Uniaxial

$$\sigma \geq \sigma_y$$

$$\sigma_1 - \sigma_3 \geq \sigma_y \quad (\sigma_1 > \sigma_2 > \sigma_3) \quad \text{or}$$

General

$$\left[\frac{1}{2} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\} \right]^{1/2} \geq \sigma_y$$

Creep

Uniaxial

$$\dot{\epsilon} = \dot{\epsilon}_o \left(\frac{\sigma}{\sigma_o} \right)^n$$

General

$$\dot{\epsilon}_1 = \dot{\epsilon}_o \frac{\sigma_o^{n-1}}{\sigma_o^n} \left\{ \sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \right\} \quad \text{etc}$$

Fracture

Uniaxial

$$\sigma \geq C K_{1c} / (\pi a)^{1/2}$$

General

$$\sigma_1 \geq C K_{1c} / (\pi a)^{1/2} \quad (\sigma_1 > \sigma_2 > \sigma_3)$$

E = Young's modulus

n = Poisson's ratio

σ_y = Plastic yield strength

n, ϵ_o , σ_o = Creep constants

K_{1c} , a = Fracture toughness, crack length

C \approx 1 Tension

C \approx 15 Compression

2. Moments of sections

A beam of uniform section, loaded in simple tension by a force F , carries a stress

$$\sigma = F/A$$

where A is the area of the section. Its response is calculated from the appropriate constitutive equation. Here the important characteristic of the section is its area, A . For other modes of loading, higher moments of the area are involved. Those for various common sections are given on the facing page. They are defined as follows.

The second moment, I , measures the resistance of the section to bending about a horizontal axis (shown as a broken line). It is

$$I = \int_{section} y^2 b(y) dy$$

where y is measured vertically and $b(y)$ is the width of the section at y . The moment K measures the resistance of the section to twisting. It is equal to the polar moment J for circular sections, where

$$J = \int_{section} 2\pi r^3 dr$$

where r is measured radially from the center of the circular section. For non-circular sections K is less than J .

The section modulus $Z = I/y_m$ (where y_m is the normal distance from the neutral axis of bending to the outer surface of the beam) measures the surface stress generated by a given bending moment, M :

$$\sigma = \frac{M y_m}{I} = \frac{M}{Z}$$

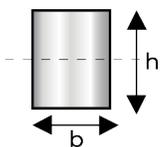
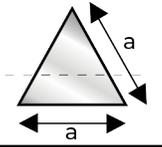
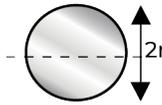
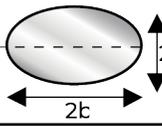
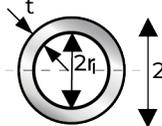
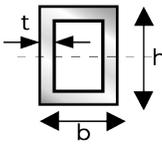
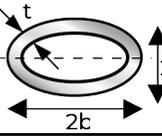
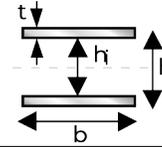
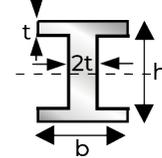
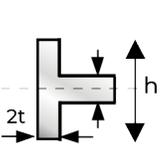
Finally, the moment Z_p , defined by

$$Z_p = \int_{section} y b(y) dy$$

measures the resistance of the beam to fully-plastic bending. The fully plastic moment for a beam in bending is

$$M_p = Z_p \sigma_y$$

2. Moments of sections.

Section shape	Area A m^2	Moment I m^4	Moment K m^4	Moment Z m^3	Moment Z _p m^3
	bh	$\frac{bh^3}{12}$	$\frac{bh^3}{3} (1 - 0.58 \frac{b}{h})$ ($h > b$)	$\frac{bh^2}{6}$	$\frac{bh^2}{4}$
	$\frac{\sqrt{3}}{4} a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{\sqrt{3}}{80} a^4$	$\frac{a^3}{32}$	$\frac{3a^3}{64}$
	πr^2	$\frac{\pi}{4} r^4$	$\frac{\pi}{2} r^4$	$\frac{\pi}{4} r^3$	$\frac{\pi}{3} r^3$
	πab	$\frac{\pi a^3 b}{4}$	$\frac{\pi a^3 b^3}{(a^2 + b^2)}$	$\frac{\pi a^2 b}{4}$	$\frac{\pi a^2 b}{3}$
	$\pi(r_o^2 - r_i^2)$ $\approx 2\pi r t$	$\frac{\pi}{4} (r_o^4 - r_i^4)$ $\approx \pi r^3 t$	$\frac{\pi}{2} (r_o^4 - r_i^4)$ $\approx 2\pi r^3 t$	$\frac{\pi}{4r_o} (r_o^4 - r_i^4)$ $\approx \pi r^2 t$	$\frac{\pi}{3} (r_o^3 - r_i^3)$ $\approx \pi r^2 t$
	$2t(h+b)$ ($h, b \gg t$)	$\frac{1}{6} h^3 t (1 + 3 \frac{b}{h})$	$\frac{2tb^2 h^2}{(h+b)} (1 - \frac{t}{h})^4$	$\frac{1}{3} h^2 t (1 + 3 \frac{b}{h})$	$bht(1 + \frac{h}{2b})$
	$\pi(a+b)t$ ($a, b \gg t$)	$\frac{\pi}{4} a^3 t (1 + \frac{3b}{a})$	$\frac{4\pi (ab)^{5/2}}{(a^2 + b^2)} t$	$\frac{\pi}{4} a^2 t (1 + \frac{3b}{a})$	$\pi abt(2 + \frac{a}{b})$
	$b(h_o - h_i)$ $\approx 2bt$ ($h, b \gg t$)	$\frac{b}{12} (h_o^3 - h_i^3)$ $\approx \frac{1}{2} bth_o^2$	--	$\frac{b}{6h_o} (h_o^3 - h_i^3)$ $\approx bth_o$	$\frac{b}{4} (h_o^2 - h_i^2)$ $\approx bth_o$
	$2t(h+b)$ ($h, b \gg t$)	$\frac{1}{6} h^3 t (1 + 3 \frac{b}{h})$	$\frac{2}{3} bt^3 (1 + 4 \frac{h}{b})$	$\frac{1}{3} h^2 t (1 + 3 \frac{b}{h})$	$bht(1 + \frac{h}{2b})$
	$2t(h+b)$ ($h, b \gg t$)	$\frac{t}{6} (h^3 + 4bt^2)$	$\frac{t^3}{3} (8b + h)$	$\frac{t}{3h} (h^3 + 4bt^2)$	$\frac{th^2}{2} \{ 1 + \frac{2t(b-2t)}{h^2} \}$

3. Elastic bending of beams

When a beam is loaded by a force F or moments M , the initially straight axis is deformed into a curve. If the beam is uniform in section and properties, long in relation to its depth and nowhere stressed beyond the elastic limit, the deflection δ , and the angle of rotation, θ , can be calculated using elastic beam theory (see Further reading in Section 16). The basic differential equation describing the curvature of the beam at a point x along its length is

$$EI \frac{d^2y}{dx^2} = M$$

where y is the lateral deflection, and M is the bending moment at the point x on the beam. E is Young's modulus and I is the second moment of area (Section 2). When M is constant this becomes

$$\frac{M}{I} = E \left(\frac{1}{R} - \frac{1}{R_0} \right)$$

where R_0 is the radius of curvature before applying the moment and R the radius after it is applied. Deflections δ and rotations θ are found by integrating these equations along the beam. The stiffness of the beam is defined by

$$S = \frac{F}{\delta} = \frac{C_1 EI}{L^3}$$

It depends on Young's modulus, E , for the material of the beam, on its length, L , and on the second moment of its section, I . The end-slope of the beam, θ , is given by

$$\theta = \frac{FL^2}{C_2 EI}$$

Equations for the deflection, δ , and end slope, θ , of beams, for various common modes of loading are shown on the facing page together with values of C_1 and C_2 .

3. Deflection of beams

	C_1	C_2
	3	2
	8	6
	2	1
	48	16
	$\frac{384}{5}$	24
	192	-
	384	-
	6	-
	-	4
	-	3

$$\delta = \frac{FL^3}{C_1EI} = \frac{ML^2}{C_1EI}$$

$$\theta = \frac{FL^2}{C_2EI} = \frac{ML}{C_2EI}$$

E = Youngs modulus (N/m^2)
 δ = Deflection (m)
 F = Force (N)
 M = Moment (Nm)
 L = Length (m)
 b = Width (m)
 t = Depth (m)
 θ = End slope (-)
 I = See Table B2 (m^4)
 y = Distance from neutral axis (m)
 R = Radius of curvature (m)

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

4. Failure of beams and panels

The longitudinal (or “fiber”) stress σ at a point y from the neutral axis of a uniform beam loaded elastically in bending by a moment M is

$$\frac{\sigma}{y} = \frac{M}{I} = E \left(\frac{1}{R} - \frac{1}{R_0} \right)$$

where I is the second moment of area (Section 2), E is Young’s modulus, R_0 is the radius of curvature before applying the moment and R is the radius after it is applied. The tensile stress in the outer fiber of such a beam is

$$\sigma = \frac{My_m}{I} = \frac{M}{Z}$$

where y_m is the perpendicular distance from the neutral axis to the outer surface of the beam. If this stress reaches the yield strength σ_y of the material of the beam, small zones of plasticity appear at the surface (top diagram, facing page). The beam is no longer elastic, and, in this sense, has failed. If, instead, the maximum fiber stress reaches the brittle fracture strength, σ_f (the “modulus of rupture”, often shortened to MOR) of the material of the beam, a crack nucleates at the surface and propagates inwards (second diagram); in this case, the beam has certainly failed. A third criterion for failure is often important: that the plastic zones penetrate through the section of the beam, linking to form a plastic hinge (third diagram).

The failure moments and failure loads, for each of these three types of failure, and for each of several geometries of loading, are given on the diagram. The formula labeled “onset” refer to the first two failure modes; those labeled “full plasticity” refer to the third. Two new functions of section shape are involved. Onset of failure involves the quantity Z ; full plasticity involves the quantity H . Both are listed in the table of Section 2, and defined in the text that accompanies it.

4. Failure of beams

	C	$M_f = Z \sigma^* \quad (\text{onset})$ $M_f = Z_p \sigma_y \quad (\text{full plasticity})$ $F_f = C Z \frac{\sigma^*}{L} \quad (\text{onset})$ $F_f = \frac{C Z_p \sigma_y}{L} \quad (\text{full plasticity})$
	1	
	1	
	2	
	4	$M_f =$ Failure moment (Nm) $F_f =$ Force at failure (N) $L =$ Length (m) $t =$ Depth (m) $b =$ Width (m) $I =$ See Table B2 (m^4) $Z =$ See Table B2 (m^3) $Z_p =$ See Table B2 (m^3) $\sigma_y =$ Yield strength (N/m^2) $\sigma_f =$ Modulus of rupture (N/m^2) $\sigma^* = \sigma_y$ (plastic material) $= \sigma_f$ (brittle material)
	8	
	8	
	16	

5. Buckling of columns, plates, and shells

If sufficiently slender, an elastic column, loaded in compression, fails by elastic buckling at a critical load, F_{crit} . This load is determined by the end constraints, of which four extreme cases are illustrated on the facing page: an end may be constrained in a position and direction; it may be free to rotate but not translate (or “sway”); it may sway without rotation; and it may both sway and rotate. Pairs of these constraints applied to the ends of column lead to the five cases shown opposite. Each is characterized by a value of the constant n that is equal to the number of half-wavelengths of the buckled shape.

The addition of the bending moment M reduces the buckling load by the amount shown in the second box. A negative value of F_{crit} means that a tensile force is necessary to prevent buckling.

An elastic foundation is one that exerts a lateral restoring pressure, p , proportional to the deflection

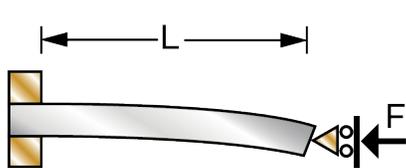
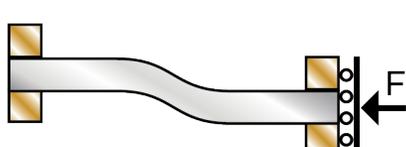
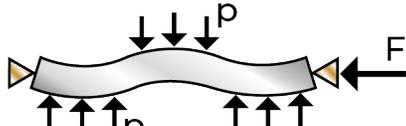
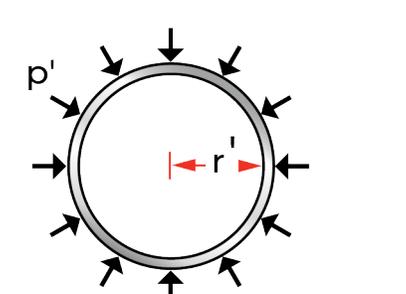
$$p = ky$$

where k is the foundation stiffness per unit depth and y the local lateral deflection. Its effect is to increase F_{crit} by the amount shown in the third box.

A thin-walled elastic tube will buckle inwards under an external pressure p' , given in the last box. Here I refers to the second moment of area of a section of the tube wall cut parallel to the tube axis.

Thin or slender shapes may buckle locally before they yield or fracture. It is this that sets a practical limit to the thinness of tube walls and webs (Chapter 9).

5. Buckling of columns, plates and shells

	n 1/2	$F_{crit} = \frac{n^2 \pi^2 EI}{L}$ <p>or</p> $\frac{F_{crit}}{A} = \frac{n^2 \pi^2 E}{(L/r)^2}$
	1	
	1	<p>F = Force (N) M = Moment (Nm) E = Young's modulus (N/m²) L = Length (m) A = Section area (m²) I = See Table B2 (m⁴) r = Gyration radius ($\sqrt{I/A}$) (m) k = Foundation stiffness (N/m²) n = Half-wavelengths in buckled shape p' = Pressure (N/m²)</p>
	3/2	
	2	
		$F_{crit} = \frac{\pi^2 EI}{L^2} - \frac{M^2}{4EI}$
		$F_{crit} = \frac{n^2 \pi^2 EI}{L^2} - \frac{k L^2}{n^2}$
		$p'_{crit} = \frac{3EI}{(r')^3}$

6. Torsion of shafts

A torque, T , applied to the ends of an isotropic bar of uniform section, and acting in the plane normal to the axis of the bar, produces an angle of twist θ . The twist is related to the torque by the first equation on the facing page, in which G is the shear modulus. For round bars and tubes of circular section, the factor K is equal to J , the polar moment of inertia of the section, defined in Section 2. For any other section shape K is less than J . Values of K are given in Section 2.

If the bar ceases to deform elastically, it is said to have failed. This will happen if the maximum surface stress exceeds either the yield strength σ_y of the material or the stress at which it fractures σ_{fr} . For circular sections, the shear stress at any point a distance r from the axis of rotation is

$$\tau = \frac{Tr}{K} = \frac{G\theta r}{K}$$

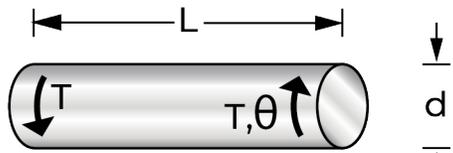
The maximum shear stress, τ_{max} , and the maximum tensile stress, σ_{max} , are at the surface and have the values

$$\tau_{max} = \sigma_{max} = \frac{Td_o}{2K} = \frac{G\theta d_o}{2L}$$

If τ_{max} exceeds $\sigma_y/2$ (using a Tresca yield criterion), or if σ_{max} exceeds, the bar fails, as shown on the figure. The maximum surface stress for the solid ellipsoidal, square, rectangular and triangular sections is at the points on the surface closest to the centroid of the section (the mid-points of the longer sides). It can be estimated approximately by inscribing the largest circle that can be contained within the section and calculating the surface stress for a circular bar of that diameter. More complex section-shapes require special consideration, and, if thin, may additionally fail by buckling.

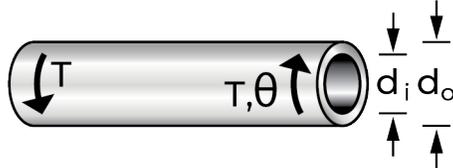
Helical springs are a special case of torsional deformation. The extension of a helical spring of n turns of radius R , under a force F , and the failure force F_{crit} , are given on the facing page.

6. Torsion of shafts



Elastic deflection

$$\theta = \frac{LT}{KG}$$

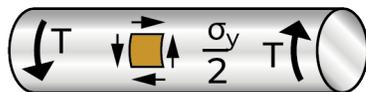


Failure

$$T_f = \frac{K \sigma_y}{d_o} \quad (\text{onset of yield})$$

$$T_f = \frac{2K \sigma_f}{d_o} \quad (\text{brittle fracture})$$

Yield



Fracture



T = Torque (Nm)

θ = Angle of twist

G = Shear modulus (N/m²)

L = Length (m)

d = Diameter (m)

K = See Table B2 (m⁴)

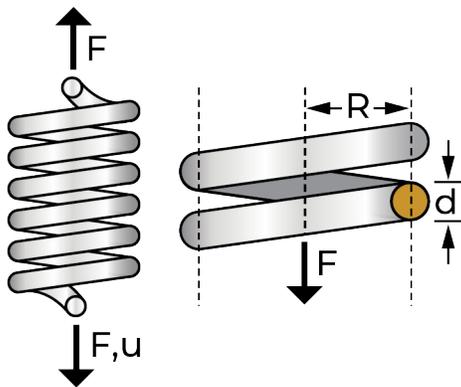
σ_y = Yield strength (N/m²)

σ_f = Modulus of rupture (N/m²)

Spring deflection and failure

$$u = \frac{64 F R^3 n}{G d^4}$$

$$F_f = \frac{\pi}{32} \frac{d^3 \sigma_y}{R}$$



F = Force (N)

u = Deflection (m)

R = Coil radius (m)

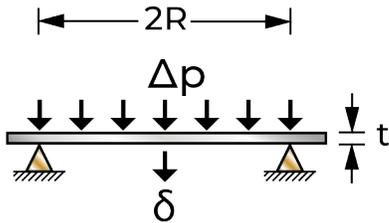
n = Number of turns

7. Static and spinning disks

A thin disk deflects when a pressure difference Δp is applied across its two surfaces. The deflection causes stresses to appear in the disk. The first box on the facing page gives deflection and maximum stress (important in predicting failure) when the edges of the disk are simply supported. The second gives the same quantities when the edges are clamped. The results for a thin horizontal disk deflecting under its own weight are found by replacing p by the mass-per-unit-area, $\rho g t$, of the disk (here ρ is the density of the material of the disk and g is the acceleration due to gravity). Thick disks are more complicated; for those, see “Further reading”.

Spinning disks, rings and cylinders store kinetic energy. Centrifugal forces generate stresses in the disk. The two boxes list the kinetic energy and the maximum stress σ_{\max} in disks and rings rotating at an angular velocity ω (radians/sec). The maximum rotation rate and energy are limited by the burst-strength of the disk. They are found by equating the maximum stress in the disk to the strength of the material.

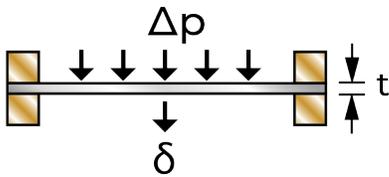
7. Static and spinning disks



Simply supported

$$\delta = \frac{3}{4} (1 - \nu^2) \frac{\Delta p R^4}{E t^3}$$

$$\sigma_{\max} = \frac{3}{8} (3 + \nu) \frac{\Delta p R^2}{t^2}$$



Clamped

$$\delta = \frac{3}{16} (1 - \nu^2) \frac{\Delta p R^4}{E t^3}$$

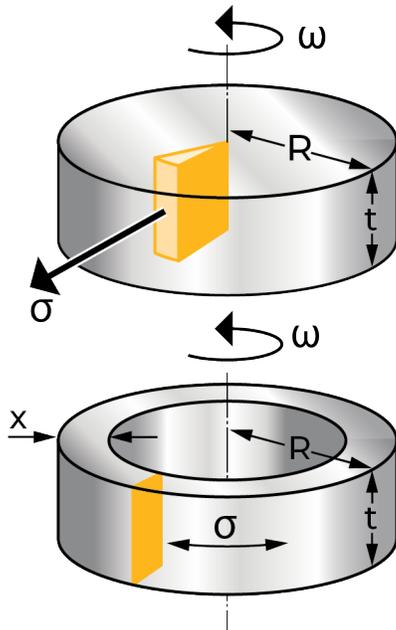
$$\sigma_{\max} = \frac{3}{8} (1 + \nu) \frac{\Delta p R^2}{t^2}$$

δ = Deflection (m)

E = Young's modulus (N/m²)

Δp = Pressure difference (N/m²)

ν = Poisson's ratio



Disk

$$U = \frac{\pi}{4} \rho t \omega^2 R^4$$

$$s_{\max} = \frac{1}{8} (3 + \nu) \rho \omega^2 R^2$$

Ring

$$U = \pi \rho t \omega^2 R^3 x$$

$$\sigma_{\max} = \rho \omega^2 R^2$$

U = Energy (J)

ω = Angular velocity (rad/s)

ρ = Density (kg/m³)

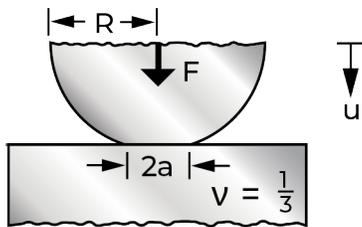
8. Contact stresses

When surfaces are placed in contact they touch at one or a few discrete points. If the surfaces are loaded, the contacts flatten elastically and the contact areas grow until failure of some sort occurs: failure by crushing (caused by the compressive stress, σ_c), tensile fracture (caused by the tensile stress, σ_t) or yielding (caused by the shear stress σ_s). The boxes on the facing page summarize the important results for the radius, a , of the contact zone, the center-to-center displacement u and the peak values of σ_c , σ_t and σ_s .

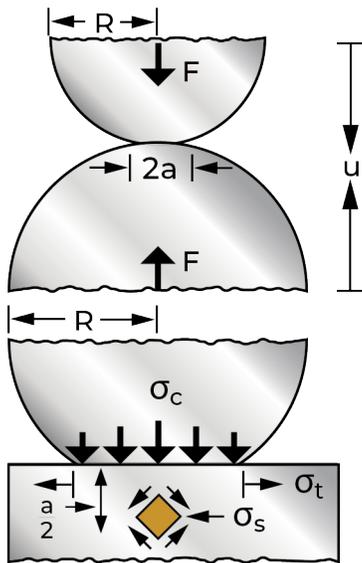
The first box shows results for a sphere on a flat, when both have the same moduli and Poisson's ratio has the value $1/3$. Results for the more general problem (the "Hertzian indentation" problem) are shown in the second box: two elastic spheres (radii R_1 and R_2 , moduli and Poisson's ratios E_1, ν_1 , and E_2, ν_2) are pressed together by a force F .

If the shear stress σ_s exceeds the shear yield strength $\sigma_y/2$, a plastic zone appears beneath the center of the contact at a depth of about $a/2$ and spreads to form the fully-plastic field shown in the two lower figures. When this state is reached, the contact pressure is approximately 3 times the yield stress, as shown in the bottom box.

8. Contact stresses



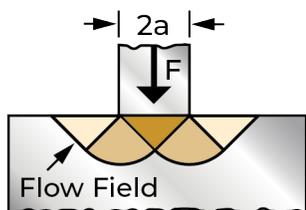
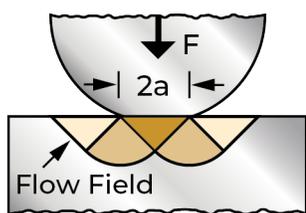
$$\left. \begin{aligned} a &= 0.7 \left(\frac{F R}{E} \right)^{\frac{1}{3}} \\ u &= 1.0 \left(\frac{F^2}{E^2 R} \right)^{\frac{1}{3}} \end{aligned} \right\} \nu = 0.33$$



$$\left. \begin{aligned} a &= \left(\frac{3}{4} \frac{F}{E^*} \frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{1}{3}} \\ u &= \left(\frac{9}{16} \frac{F^2}{(E^*)^2} \frac{R_1 + R_2}{R_1 R_2} \right)^{\frac{1}{3}} \end{aligned} \right\}$$

$$\begin{aligned} (\sigma_c)_{\max} &= \frac{3F}{2\pi a^2} \\ (\sigma_s)_{\max} &= \frac{F}{2\pi a^2} \\ (\sigma_t)_{\max} &= \frac{F}{6\pi a^2} \end{aligned}$$

- R_1, R_2 = Radii of spheres (m)
- E_1, E_2 = Moduli of spheres (N/m^2)
- ν_1, ν_2 = Poisson's ratio
- F = Load (N)
- a = Radius of contact (m)
- u = Displacement (m)
- σ = Stresses (N/m^2)
- σ_y = Yield stress (N/m^2)
- $E^* = \left(\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)^{-1}$



$$\frac{F}{\pi a^2} = 3\sigma_y$$

9. Estimates for stress concentrations

Stresses and strains are concentrated at holes, slots or changes of section in elastic bodies. Plastic flow, fracture and fatigue cracking start at these places. The local stresses at the stress concentrations can be computed numerically, but this is often unnecessary. Instead, they can be estimated using the equation shown on the facing page.

The stress concentration caused by a change in section dies away at distances of the order of the characteristic dimension of the section-change (defined more fully below), an example of St Venant's principle at work. This means that the maximum local stresses in a structure can be found by determining the nominal stress distribution, neglecting local discontinuities (such as holes or grooves), and then multiplying the nominal stress by a stress concentration factor. Elastic stress concentration factors are given approximately by the equation. In it, σ_{nom} is defined as the load divided by the minimum cross-section of the part, ρ is the minimum radius of curvature of the stress-concentrating groove or hole, and c is the characteristic dimension: either the half-thickness of the remaining ligament, the half-length of a contained crack, the length of an edge-crack or the height of a shoulder, whichever is least. The drawings show examples of each such situation. The factor α is roughly 2 for tension, but is nearer 1/2 for torsion and bending. Though inexact, the equation is an adequate working approximation for many design problems.

The maximum stress is limited by plastic flow or fracture. When plastic flow starts, the strain concentration grows rapidly while the stress concentration remains constant. The strain concentration becomes the more important quantity, and may not die out rapidly with distance (St Venant's principle no longer applies).

9. Stress concentrations

$$\frac{\sigma_{\max}}{\sigma_{\text{nom}}} = 1 + a \left(\frac{c}{r} \right)^{\frac{1}{2}}$$

F = Force (N)
A_{min} = Minimum section (m²)
σ_{nom} = F/A_{min} (N/m²)
r = Radius of curvature (m)
c = Characteristic length (m)
a ≈ 0.5 (Torsion)
a ≈ 2.0 (Tension)

10. Sharp cracks

Sharp cracks (that is, stress concentrations with a tip radius of curvature of atomic dimensions) concentrate stress in an elastic body more acutely than rounded stress concentrations do. To a first approximation, the local stress falls off as $1/r^{1/2}$ with radial distance r from the crack tip. A tensile stress σ , applied normal to the plane of a crack of length $2a$ contained in an infinite plate (as in the top figure on the facing page) gives rise to a local stress field σ_l that is tensile in the plane containing the crack and given by

$$\sigma_l = \frac{C\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}$$

where r is measured from the crack tip in the plane $\theta = 0$ and C is a constant. The mode 1 stress intensity factor K_I , is defined as

$$K_I = C\sigma\sqrt{\pi a}$$

Values of the constant C for various modes of loading are given on the figure. The stress σ for point loads and moments is given by the equations at the bottom. The crack propagates when $K_I > K_{Ic}$, the fracture toughness.

When the crack length is very small compared with all specimen dimensions and compared with the distance over which the applied stress varies, C is equal to 1 for a contained crack and 1.1 for an edge crack. As the crack extends in a uniformly loaded component, it interacts with the free surfaces, giving the correction factors shown opposite. If, in addition, the stress field is non-uniform (as it is in an elastically bent beam), C differs from 1; two examples are given on the figure. The factors, C , given here, are approximate only, good when the crack is short but not when the crack tips are very close to the boundaries of the sample. They are adequate for most design calculations. More accurate approximations, and other less common loading geometries can be found in the references listed in "Further reading".

10. Sharp cracks

	<p>C</p> <p>1.0 if $a \ll w$</p> $\left(\frac{2w}{\pi a} \tan \frac{\pi a}{2w} \right)^{\frac{1}{2}}$	$K_I = C \sigma \sqrt{\pi a}$ <p>Failure when</p> $K_I \geq K_{Ic}$
	<p>1.1 if $a \ll w$</p> $\left(\frac{2w}{\pi a} \tan \frac{\pi a}{2w} \right)^{\frac{1}{2}} \frac{1}{1 - \frac{a}{w}}$	<p>K_I = Stress intensity ($\text{N}/\sqrt{\text{m}}$) σ = Remote stress (N/m^2) F = Load (N) M = Moment (Nm) a = Internal crack half-length (m) = Surface crack length (m) w = Half-width (centre) (m) = Width (edge crack) (m) b = Sample depth (m) t = Beam thickness (m)</p>
	<p>1.1 if $a \ll w$</p> $\frac{1.1 \left(1 - 0.2 \frac{a}{w} \right)}{\left(1 - \frac{a}{w} \right)^{\frac{3}{2}}}$	
	$\frac{2}{\pi} \left(\frac{a+c}{a-c} \right)^{\frac{1}{2}}$	
	<p>1.1</p> $\frac{\left(1 - \frac{3}{2} \frac{a}{t} \right)}{\left(1 - \frac{a}{t} \right)^{\frac{3}{2}}}$	<p>Moment on beam</p> $\sigma = \frac{6 M}{bt}$
	<p>1.1</p> $\frac{\left(1 - \frac{3}{2} \frac{a}{t} \right)}{\left(1 - \frac{a}{t} \right)^{\frac{3}{2}}}$	<p>3 - point bending</p> $\sigma = \frac{3 FL}{2 bt}$

11. Pressure vessels

Thin-walled pressure vessels are treated as membranes. The approximation is reasonable when $t < b/4$. The stresses in the wall are given on the facing page; they do not vary significantly with radial distance, r . Those in the plane tangent to the skin, σ_θ and σ_z for the cylinder and σ_θ and σ_ϕ for the sphere, are just equal to the internal pressure amplified by the ratio b/t or $b/2t$, depending on geometry. The radial stress σ_r is equal to the mean of the internal and external stress, $p/2$ in this case. The equations describe the stresses when an external pressure p_e is superimposed if p is replaced by $(p - p_e)$.

In thick-walled vessels, the stresses vary with radial distance r from the inner to the outer surfaces, and are greatest at the inner surface. The equations can be adapted for the case of both internal and external pressures by noting that when the internal and external pressures are equal, the state of stress in the wall is

$$\sigma_\theta = \sigma_r = -p \text{ (cylinder)}$$

or
$$\sigma_\theta = \sigma_\phi = \sigma_r = -p \text{ (sphere)}$$

allowing the term involving the external pressure to be evaluated. It is not valid to just replace p by $(p - p_e)$.

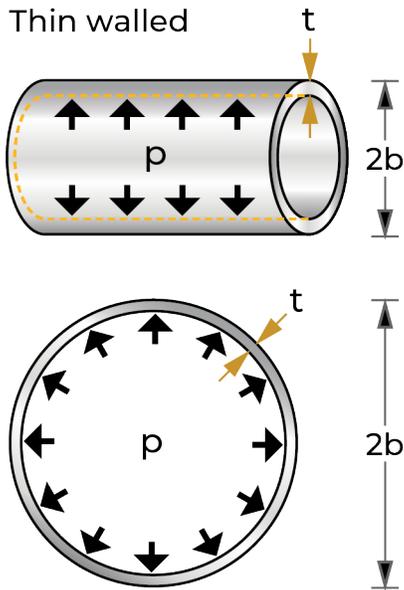
Pressure vessels fail by yielding when the Von Mises equivalent stress first exceed the yield strength, σ_y . They fail by fracture if the largest tensile stress exceeds the fracture stress σ_{fr} , where

$$\sigma_{fr} = \frac{CK_{1c}}{\sqrt{\pi a}}$$

and K_{1c} is the fracture toughness, a the half-crack length and C a constant given in Section 10.

11. Pressure vessels

Thin walled



Cylinder

$$\sigma_{\theta} = \frac{p b}{t}$$

$$\sigma_r = -\frac{p}{2}$$

$$\sigma_z = \frac{p b}{2 t} \quad (\text{Closed ends})$$

Sphere

$$\sigma_{\theta} = \sigma_{\phi} = \frac{p b}{2 t}$$

$$\sigma_r = -\frac{p}{2}$$

p = Pressure (N/m²)

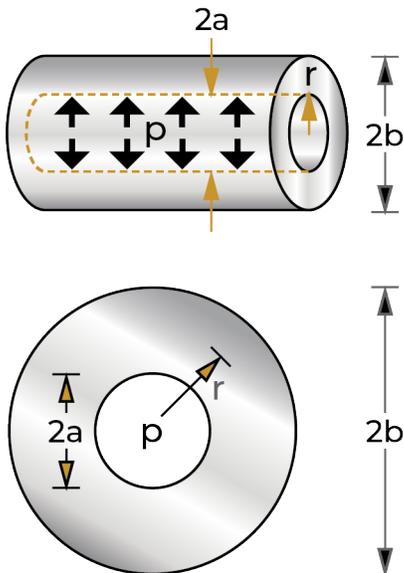
t = Wall thickness (m)

a = Inner radius (m)

b = Outer radius (m)

r = Radial coordinate (m)

Thick walled



Cylinder

$$\sigma_{\theta} = \frac{p a^2}{r^2} \left(\frac{b^2 - r^2}{b^2 - a^2} \right)$$

$$\sigma_r = -\frac{p a^2}{r^2} \left(\frac{b^2 + r^2}{b^2 - a^2} \right)$$

Sphere

$$\sigma_{\theta} = \sigma_{\phi} = \frac{p a^3}{2 r^3} \left(\frac{b^3 + 2 r^3}{b^3 - a^3} \right)$$

$$\sigma_r = -\frac{p a^3}{r^3} \left(\frac{b^3 - r^3}{b^3 - a^3} \right)$$

12. Vibrating beams, tubes and disks

Any undamped system vibrating at one of its natural frequencies can be reduced to the simple problem of a mass m attached to a spring of stiffness K . The lowest natural frequency of such a system is

$$f = \frac{1}{2\pi} \sqrt{\frac{K}{m}}$$

Specific cases require specific values for m and K . They can often be estimated with sufficient accuracy to be useful in approximate modeling. Higher natural frequencies are simple multiples of the lowest.

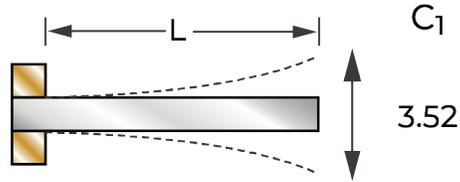
The first box on the facing page gives the lowest natural frequencies of the flexural modes of uniform beams with various end-constraints. As an example, the first can be estimated by assuming that the effective mass of the beam is one quarter of its real mass, so that

$$m = \frac{m_0 L}{4}$$

where m_0 is the mass per unit length of the beam and that K is the bending stiffness (given by F/δ from Section 3); the estimate differs from the exact value by 2%. Vibrations of a tube have a similar form, using I and m_0 for the tube. Circumferential vibrations can be found approximately by “unwrapping” the tube and treating it as a vibrating plate, simply supported at two of its four edges.

The second box gives the lowest natural frequencies for flat circular disks with simply-supported and clamped edges. Disks with doubly-curved faces are stiffer and have higher natural frequencies.

12. Vibrating beams, tubes and disks

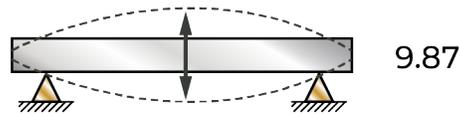


C_1

3.52

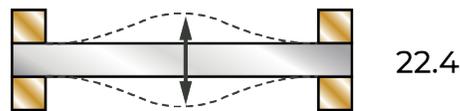
Beams and tubes

$$f = \frac{C_1}{2\pi} \sqrt{\frac{EI}{m_o L^4}}$$

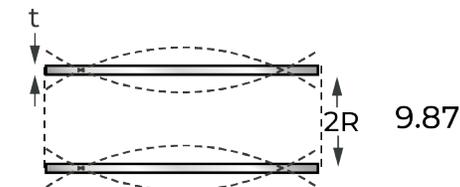


9.87

f = Natural frequency (s^{-1})
 m_o = ρA = Mass / length (kg/m)
 ρ = Density (kg/m^3)
 A = Section area (m^2)
 I = See Table B2

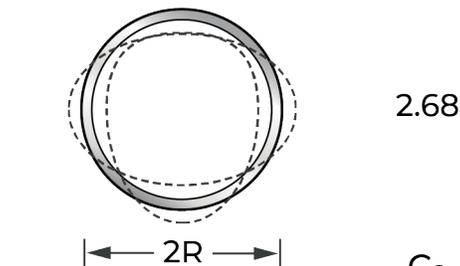


22.4



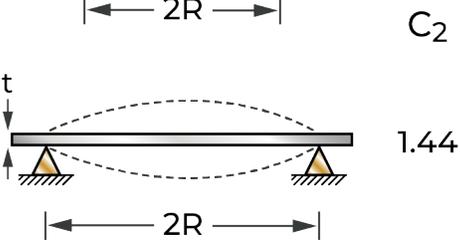
9.87

$$\left\{ \begin{array}{l} \text{With } A = 2\pi R t \\ I = \pi R^3 t \end{array} \right.$$



2.68

$$\left\{ \begin{array}{l} \text{With } A = L t \\ I = \frac{L t^3}{12} \end{array} \right. \quad L = \text{Length of tube}$$

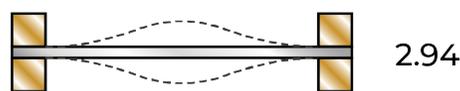


1.44

C_2

Disks

$$f = \frac{C_2}{2\pi} \sqrt{\frac{E t^3}{m_1 R^4 (1 - \nu^2)}}$$



2.94

m_1 = ρt = Mass / area (kg/m^2)
 t = Thickness (m)
 R = Radius (m)
 ν = Poisson's ratio

13. Creep and creep fracture

At temperatures above $1/3 T_m$ (where T_m is the absolute melting point), materials creep when loaded. It is convenient to characterize the creep of a material by its behavior under a tensile stress σ , at a temperature T_m . Under these conditions the steady-state tensile strain rate $\dot{\epsilon}_{ss}$ is often found to vary as a power of the stress and exponentially with temperature:

$$\dot{\epsilon}_{ss} = A \left(\frac{\sigma}{\sigma_0} \right)^n \exp \left(-\frac{Q}{RT} \right)$$

where Q is an activation energy, A is a kinetic constant and R is the gas constant. At constant temperature this becomes

$$\dot{\epsilon}_{ss} = \dot{\epsilon}_0 \left(\frac{\sigma}{\sigma_0} \right)^n$$

where $\dot{\epsilon}_0$ (s^{-1}), σ_0 (N/m^2) and n are creep constants.

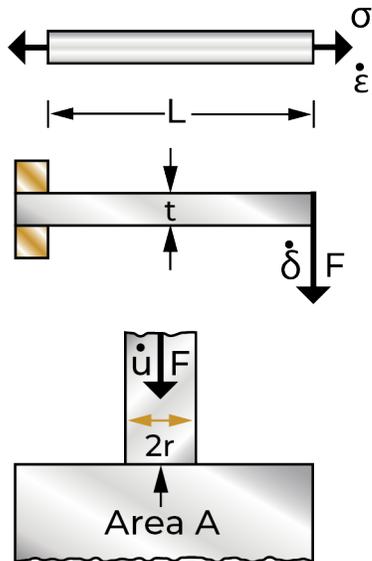
The behavior of creeping components is summarized on the facing page that give the deflection rate of a beam, the displacement rate of an indenter and the change in relative density of cylindrical and spherical pressure vessels in terms of the tensile creep constants.

Prolonged creep causes the accumulation of creep damage that ultimately leads, after a time t_f , to fracture. To a useful approximation

$$t_f \dot{\epsilon}_{ss} = C$$

where C is a constant characteristic of the material. Creep-ductile material have values of C between 0.1 and 0.5; creep-brittle materials have values of C as low as 0.01.

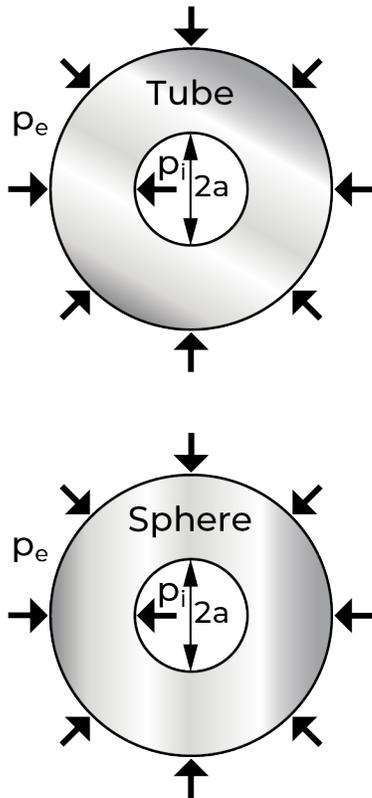
13. Creep



$$\dot{\epsilon} = \dot{\epsilon}_o \left(\frac{\sigma}{\sigma_o} \right)^n$$

$$\dot{\delta} = \frac{2 \dot{\epsilon}_o}{n+2} \left(\frac{(2n+1) F}{n \sigma_o b t} \right)^n \left(\frac{L}{t} \right)^{n+1}$$

$$\dot{u} = C_1 \dot{\epsilon}_o \sqrt{A} \left(\frac{C_2 F}{\sigma_o A} \right)^n$$



σ = Stress (N/m²)

F = Force (N)

$\dot{\delta}, \dot{u}$ = Displacement rates (m/s)

$n, \dot{\epsilon}_o, \sigma_o$ = Creep constants

L, b, t = Beam dimensions (m)

a, b = Radii of pressure vessels (m)

ρ = Relative density, $\frac{b^3 - a^3}{b^3}$

$\dot{\rho}$ = Rate of change of ρ

C_1, C_2 = Constants

$$\dot{\rho} = 2 \dot{\epsilon}_o \frac{\rho (1 - \rho)}{(1 - (1 - \rho)^{1/n})^n} \left(\frac{2 (p_e - p_i)}{n \sigma_o} \right)^n$$

$$\dot{\rho} = \frac{3}{2} \dot{\epsilon}_o \frac{\rho (1 - \rho)}{(1 - (1 - \rho)^{1/n})^n} \left(\frac{3 (p_e - p_i)}{2n \sigma_o} \right)^n$$

14. Flow of heat and matter

Heat flow can be limited by conduction, by convection or by radiation. The constitutive equations for each are listed on the facing page. The first equation is Fourier's first law, describing steady-state heat flow; it contains the thermal conductivity, λ . The second is Fourier's second law, which treats transient heat-flow problems; it contains the thermal diffusivity, a , defined by

$$a = \frac{\lambda}{\rho C}$$

where ρ is the density and C the specific heat at constant pressure. Solutions to these two differential equations are given in Section 15.

The third equation describes convective heat transfer. It, rather than conduction, limits heat flow when the Biot number

$$B_i = \frac{hs}{\lambda} < 1$$

where h is the heat-transfer coefficient and s is a characteristic dimension of the sample. When, instead, $B_i > 1$, heat flow is limited by conduction. The final equation is the Stefan-Boltzmann law for radiative heat transfer. The emissivity, ϵ , is unity for black bodies; less for all other surfaces.

Diffusion of matter follows a pair of differential equations with the same form as Fourier's two laws, and with similar solutions. They are commonly written

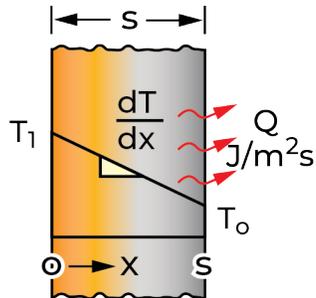
$$J = -D\nabla C = -D \frac{dC}{dx} \text{ (steady state)}$$

and

$$\frac{dC}{dt} = D\nabla^2 C = D \frac{d^2C}{dx^2} \text{ (time-dependent flow)}$$

where J is the flux, C is the concentration, x is the distance and t is time. Solutions are given in Section 15.

14. Constitutive equations for heat flow



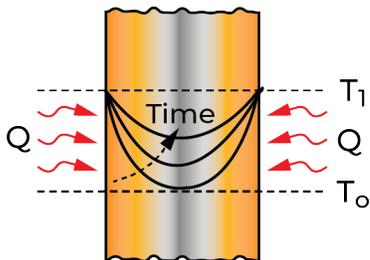
$$Q = -\lambda \nabla T = -\lambda \frac{dT}{dx}$$

Q = Heat flux (J/m²s)

T = Temperature (K)

x = Distance (m)

λ = Thermal conductivity (W/mK)



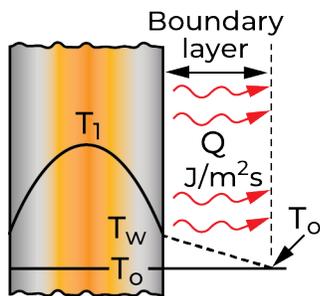
$$\frac{\partial T}{\partial t} = a \nabla^2 T = a \frac{\partial^2 T}{\partial x^2}$$

t = Time (s)

ρ = Density (kg/m³)

C_p = Specific heat (J/kg.K)

a = Thermal diffusivity, $\frac{\lambda}{\rho C_p}$ (m²/s)



$$Q = h (T_w - T_o)$$

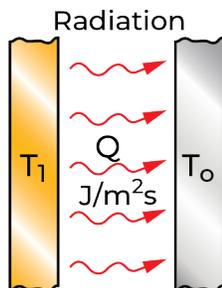
T_w = Surface temperature (K)

T = Fluid temperature (K)

h = Heat transfer coefficient (W/m²K)

= 5 - 50 W/m².K in air

= 1000 - 5000 W/m².K in water



$$Q = \epsilon \sigma (T_1^4 - T_0^4)$$

ϵ = Emissivity (1 for black body)

σ = Stefan's constant

= 5.67×10^{-8} W/m².K⁴

15. Solutions for diffusion equations

Solutions exist for the diffusion equations for a number of standard geometries. They are worth knowing because many real problems can be approximated by one of these.

At steady-state the temperature or concentration profile does not change with time. This is expressed by the boxed equations within the first box at the top of the facing page. Solutions for these are given below for uniaxial flow, radial flow in a cylinder and radial flow in a sphere. The solutions are fitted to individual cases by matching the constants A and B to the boundary conditions. Solutions for matter flow are found by replacing temperature, T, by concentration, C, and conductivity, λ , by diffusion coefficient, D.

The box within the second large box summarizes the governing equations for time-dependent flow, assuming that the diffusivity (a or D) is not a function of position. Solutions for the temperature or concentration profiles, T(x,t) or C(x,t), are given below. The first equation gives the “thin-film” solution: a thin slab at temperature T_1 , or concentration C_1 is sandwiched between two semi-infinite blocks at T_0 or C_0 , at $t = 0$, and flow allowed. The second result is for two semi-infinite blocks, initially at T_1 and T_0 , (or C_1 or C_0) brought together at $t = 0$. The last is for a T or C profile that is sinusoidal, of wavelength $\lambda/2\pi$ and amplitude A at $t = 0$.

Note that all transient problems end up with a characteristic time constant t^* with

$$t^* = \frac{x^2}{\beta a} \text{ or } \frac{x^2}{\beta D}$$

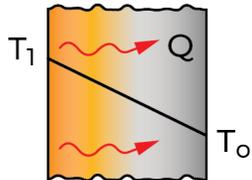
where x is a dimension of the specimen; or a characteristic length x^* with

$$x^* = \sqrt{\beta a t} \text{ or } \sqrt{\beta D t}$$

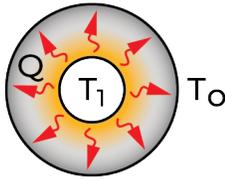
where t is the time scale of observation, with $1 < \beta < 4$, depending on geometry.

15. Solutions for diffusion equations

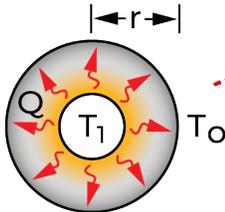
(1-D)



(2-D)



(3-D)



Steady state

$$\lambda \nabla^2 T = 0$$

$$D \nabla^2 C = 0$$

$$\lambda T(x) = Ax + B \quad (1-D)$$

$$\lambda T(r) = A \ln r + B \quad (2-D)$$

$$\lambda T(r) = \frac{A}{r} + B \quad (3-D)$$

λ = Thermal conductivity (W/m.K)

D = Diffusion coefficient (m²/s)

C = Concentration

A, B = Constants of Integration

x = Linear distance (m)

r = Radial distance (m)

a = Thermal diffusivity (m²/s)

Transient

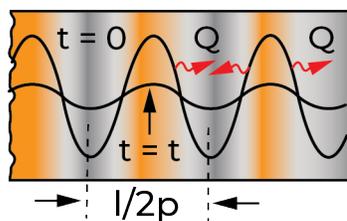
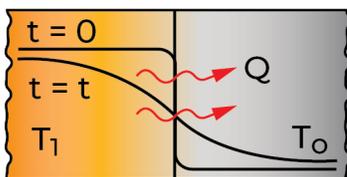
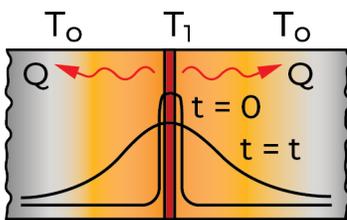
$$\frac{\partial T}{\partial t} = a \nabla^2 T$$

$$\frac{\partial C}{\partial t} = D \nabla^2 C$$

$$T(x,t) = \frac{A}{\sqrt{at}} \exp\left(-\frac{x^2}{4at}\right) + B$$

$$T(x,t) = A \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right)\right) + B$$

$$T(x,t) = A \sin(\lambda x) \exp(-\lambda^2 at)$$



16. Thermal field for moving heat source

When a plate is exposed to a heat source q moving at velocity v , a thermal field is created. The field tracks along with the moving heat source such the temperature T at a distance r from the path of the source first rises, then decays back to ambient temperature T_o . In welding, the peak temperature T_p reached in the cycle exceeds the melting point; in brazing and soldering it only reaches the melting point of the braze or solder.

Three characteristics of the thermal field are useful: the temperature T as a function of position r and time t ; the peak temperature T_p reached at the point r during the thermal cycle; and (for steels) the time interval Δt to cool from 800 to 500 C because this determines the structure of the weld at that point. For a thin plate (one with a thickness d that is small compared to the width of the thermal field) these quantities are given by

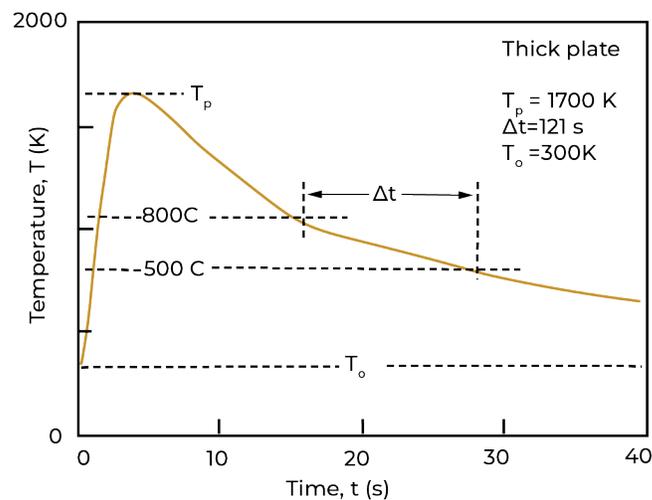
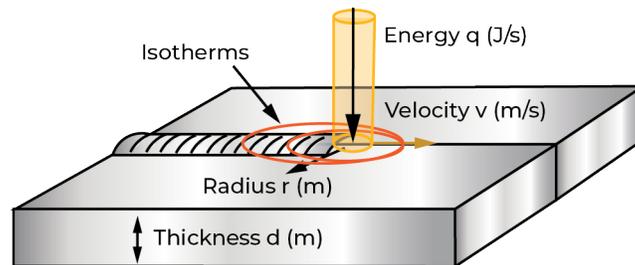
$$T = T_o + \frac{q/v}{d(4\pi\lambda\rho C_p t)^{1/2}} \exp - \left(\frac{r^2}{4at} \right) \quad T_p = T_o + \left(\frac{2}{\pi e} \right)^{1/2} \frac{q/v}{2d\rho C_p r} \quad \Delta t = \frac{(q/v d)^2}{4.6 \times 10^6 \lambda \rho C_p}$$

For a thick plate (one with a thickness that is large compared to the width of the thermal field) the corresponding equations are

$$T = T_o + \frac{q/v}{2\pi\lambda t} \exp - \left(\frac{r^2}{4at} \right) \quad T_p = T_o + \left(\frac{2}{\pi e} \right) \frac{q/v}{\rho C_p r^2} \quad \Delta t = \frac{q/v}{1.2 \times 10^3 \lambda}$$

Symbols are defined on the facing page.

16. Temperature field of a moving heat source



- q = Energy input from beam (J/s)
- v = Beam tracking velocity (m/s)
- r = Lateral distance from beam center (m)
- d = Plate thickness (m)
- T = Temperature (K)
- T_p = Peak temperature (K)
- T_o = Temperature remote from source (K)
- t = Time (s)
- Δt = Time for temperature to fall from 800 to 500C (s)
- λ = Thermal conductivity (W/m.K)
- ρ = Density (kg/m³)
- C_p = Specific heat (J/kg.K)
- $a = \lambda/\rho C_p$ = Thermal diffusivity (m²/s)
- e = Base of natural logarithms (2.718)

17. Further reading

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Conversion of units--stress and pressure

To →	MPa	dyn/cm ²	lb/in ²	kgf/mm ²	bar	long ton/in ²
From ↓	Multiply by					
MPa	1	10 ⁷	1.45 x 10 ²	0.102	10	6.48 x 10 ⁻²
dyn/cm ²	10 ⁻⁷	1	1.45 x 10 ⁻⁵	1.02 x 10 ⁻⁸	10 ⁻⁶	6.48 x 10 ⁻⁹
lb/in ²	6.89 x 10 ⁻³	6.89 x 10 ⁴	1	7.03 x 10 ⁻⁴	6.89 x 10 ⁻²	4.46 x 10 ⁻⁴
kgf/mm ²	9.81	9.81 x 10 ⁷	1.42 x 10 ³	1	98.1	63.5 x 10 ⁻²
bar	0.10	10 ⁶	14.48	1.02 x 10 ⁻²	1	6.48 x 10 ⁻³
long ton/in ²	15.44	1.54 x 10 ⁸	2.24 x 10 ³	1.54	1.54 x 10 ²	1

Conversion of units – energy*

To →	MJ	kWhr	kcal	Btu	ft lbf	toe
From ↓	Multiply by					
MJ	1	0.278	239	948	0.738 x 10 ⁶	23.8 x 10 ⁻⁶
kWhr	3.6	1	860	3.41 x 10 ³	2.66 x 10 ⁶	85.7 x 10 ⁻⁶
kcal	4.18 x 10 ⁻³	1.16 x 10 ⁻³	1	3.97	3.09 x 10 ³	99.5 x 10 ⁻⁹
Btu	1.06 x 10 ⁻³	0.293 x 10 ⁻³	0.252	1	0.778 x 10 ³	25.2 x 10 ⁻⁹
ft lbf	1.36 x 10 ⁻⁶	0.378 x 10 ⁻⁶	0.324 x 10 ⁻³	1.29 x 10 ⁻³	1	32.4 x 10 ⁻¹²
toe	41.9 x 10 ³	11.6 x 10 ³	10 x 10 ⁶	39.7 x 10 ⁶	30.8 x 10 ⁹	1

*MJ = megajoules; kWhr = kilowatt hour; kcal = kilocalorie; Btu = British thermal unit; ft lbf = foot-pound force; toe = tonnes oil equivalent.

Conversion of units – power*

To →	kW (kJ/s)	kcal/s	hp	ft lbf/s	Btu/h
From ↓	Multiply by				
kW (kJ/s)	1	4.18	1.34	735	4.47 x 10 ⁴
kcal/s	0.239	1	0.321	176	1.07 x 10 ⁴
hp	0.746	3.12	1	550	3.35 x 10 ⁴
ft lbf/s	1.36 x 10 ⁻³	5.68 x 10 ⁻³	1.82 x 10 ⁻³	1	
Btu/h	2.24 x 10 ⁻⁵	9.33 x 10 ⁻⁵	3.0 x 10 ⁻⁵		1

*kW = kilowatt; kcal/s = kilocalories per second; hp = horse power; ft lb/s = foot-pounds/second; Btu/h = British thermal units/hour

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