ALTERNATIVE CONVERGENCE-DIVERGENCE CHECKS FOR STRESSES FROM FEA

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How Many Stress Singularities Can You Find?

• **Problem 1: Bicycle Wrench**
  - In brief, this problem asks the stress analyst to use ANSYS to find the von Mises stresses for the bike wrench shown below.

  ![Bicycle Wrench Diagram](image)

  - 2D stress analysis problem from Moaveni, 2003
How Many Stress Singularities Can You Find?

• Answers for Problem 1
  – There is a single stress singularity at the points marked with S, two stress singularities at the points marked with S2.

  – Thus there are, in total, 32 stress singularities in this problem.
How Many Stress Singularities Can You Find?

- **Problem 2: Tooth Implant**
  - In brief, this problem asks the stress analyst to use FEA to find the maximum principal stresses for the model of a tooth implant shown below.

  - 2D stress analysis problem from Logan (2007)

  - E = 1.6 x 10^6 psi for implant (cross-hatched)
  - E = 1.0 x 10^6 psi for bone
• **Answers for Problem 2**
  - There is a single stress singularity at the points marked with S, two stress singularities at the points marked with S2.
  - Thus there are, in total, 12 stress singularities in this problem.
• **Problem 3: Plate with Reinforced Hole**
  - In brief, this problem asks the stress analyst to use ANSYS to find the stresses (including the maximum values) for the plate shown below.

  2D stress analysis problem from Madenci & Guven, 2006
• **Answers for Problem 3**
  - There is a single stress singularity at the points marked with S.
  - Thus there are, in total, 6 stress singularities in this problem.
How Many Stress Singularities Can You Find?

• Problem 4: Loaded Bar Fixed at One End
  – In brief, this problem asks the stress analyst to use FEA to find the maximum principal stress for the loaded-clamped bar below (Poisson’s ratio = 0.26)
  – 3D stress analysis problem from Zhang, 2005
• **Answers for Problem 4**
  - There are line singularities at positions marked with $SL$ and point singularities at positions marked with $SP$.
  - Thus there are, in total, 8 stress singularities in this problem.
Stress Singularities Are the Bane of Stress Analysis with FEA

• Stress singularities **occur frequently** in stress analysis, and more frequently in 3D than in 2D. By way of example, all of the following recent FEA texts contain examples/problems with stress singularities (with no mention of this being the case): Akin, 2005; Bhatti, 2005; Logan, 2007; Madenci & Guven, 2006; Moaveni, 2003; Reddy, 2006; Zhang, 2005; and Zienkiewicz et al., 2005. In total, more than 30 examples with stress singularities and more than 50 problems with stress singularities are contained in these texts.

• Trying to use FEA to **compute** the stress at a singularity is an exercise in **futility**.

• Trying to use the FEA stress computed at a singularity in **stress-strength comparisons** is an exercise in **foolishness**.
Detected Stress Singularities in Stress Analysis

• To avoid exercises in futility/foolishness it is crucial that the stress analyst be aware of the presence of every stress singularity in a configuration being subjected to FEA.

• There are two methods of detection: asymptotic identification and convergence-divergence checks.

• **Asymptotic identification**: Sinclair, 2004, gives a review of stress singularities identified asymptotically in the literature.

• **Convergence-divergence checks**: The method we pursue here next.
Convergence-Divergence Checks: Mesh Refinement

- Minimum of **three** meshes required: $m = 1, 2, 3, \ldots$
- Meshes formed by **successive scaling** of element sides
- In **2D**, typically halve element sides, e.g.

<table>
<thead>
<tr>
<th>Mesh No., $m$</th>
<th>Initial mesh</th>
<th>First refinement</th>
<th>Second refinement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$N$</td>
<td>$4N$</td>
<td>$16N$</td>
</tr>
</tbody>
</table>

- In **3D**, halving may prove computationally too challenging. Instead can use a reduced scale factor/mesh coarsening, e.g., with a scale factor $\sim 1\frac{1}{2}$, the following mesh sequence can result:

<table>
<thead>
<tr>
<th>Mesh No., $m$</th>
<th>First coarsened mesh</th>
<th>Initial mesh</th>
<th>First refinement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>No. of elements</td>
<td>$0.3N$</td>
<td>$N$</td>
<td>$3.4N$</td>
</tr>
</tbody>
</table>
Baseline (10%) Convergence-Divergence Checks†

• Converging vs. Divergence
  – Suppose $\sigma$ is the stress component of interest at the location of concern in an application. Denote its value as found by FEA on mesh $m$ by $\sigma_m (m = 1, 2, 3)$. Then the FEA for $\sigma$ is judged to be converging if changes in its value as mesh refinement proceeds decrease by more than 10%. Otherwise the FEA is judged to exhibit divergence. That is
    
    $|\sigma_1 - \sigma_2| > 1.1 |\sigma_2 - \sigma_3|$ \rightarrow converging \hspace{1cm} (1)
    
    $|\sigma_1 - \sigma_2| \leq 1.1 |\sigma_2 - \sigma_3|$ \rightarrow divergence \hspace{1cm} (2)

• Case 1: Converging Stresses
  – If (1) holds, we need to decide if the FEA for $\sigma$ has converged. We judge $\sigma_3$ to have converged to within a satisfactory level, a good level, or an excellent level if the error estimate for $\sigma_3$, $\hat{e}$, is below 10%, 5%, or 1%, respectively. That is
    
    $5 \leq \hat{e} \ (%) < 10$ \rightarrow satisfactory
    
    $1 \leq \hat{e} \ (%) < 5$ \rightarrow good \hspace{1cm} (3)
    
    $\hat{e} \ (%) < 1$ \rightarrow excellent
  – Then we accept $\sigma_3$ as our FEA determination of $\sigma$ within the applicable error level.

† These baseline checks are essentially the same as those presented at the last ANSYS Conference: Sinclair, Beisheim and Sezer, 2006.
Baseline (10%) Convergence-Divergence Checks Cont’d

- **Error Estimate**

\[ \hat{e} = \left( \frac{\sigma_2 - \sigma_3}{\Gamma} \right) \text{ (x100 for %)} \] \hspace{1cm} (4)

- In (4), \( \Gamma \) is a factor reflecting the rate of convergence of the FEA. If convergence is slow, \( \Gamma < 1 \) and \( \hat{e} \) is increased by \( \Gamma \)'s presence: If convergence is fast, \( \Gamma > 1 \) and \( \hat{e} \) is decreased. More precisely

\[ \Gamma = 2^\hat{c} - \left| \frac{\sigma_2}{\sigma_3} \right| \] \hspace{1cm} (5)

- The expression for \( \Gamma \) in (5) owes its origin to verification research in the CFD community (see Roache, 1998, Ch. 5). In (5), \( \hat{c} \) is an estimate of the actual convergence rate of the FEA and is given by

\[ \hat{c} = \frac{\ln \left( \frac{\sigma_1 - \sigma_2}{\sigma_2 - \sigma_3} \right)}{\ln 2} \] \hspace{1cm} (6)
Baseline (10%) Convergence-Divergence Checks Cont’d

• **Case 2: Stresses Exhibiting Divergence**
  - If (2) holds, we need to decide if the FEA for $\sigma$ is really diverging because of the presence of a stress singularity or just not yet converged with $\sigma_3$ being well short of its true and typically relatively large value. We judge the FEA to be diverging if one of the following singularity signatures is complied with: Then further FEA is pointless. Otherwise we judge the FEA to not be converged: Then further FEA may yet yield accurate results.

• **Power Singularities**
  - If the stress $\sigma$ has a power singularity then
    
    $\sigma = O(r^{-\gamma})$ as $r \to 0$ \hspace{1cm} (7)
  
  where $r$ is the distance from the singularity at the location of concern, and $\gamma$ is the singularity exponent. When (2) holds, $\hat{c}$ of (6) becomes negative and furnishes an estimate $\hat{\gamma}$ of the singularity exponent. That is
    
    $\hat{\gamma} = -\hat{c}$ \hspace{1cm} (8)
• **Power Singularities** (cont’d)
  
  - If successive estimates of $\hat{\gamma}$ differ by less than 10% of their mean value, we judge a power singularity to be present. This judgment does require one more mesh to be run ($m = 4$). Let $\hat{\gamma}_3$ be the singularity exponent estimate from (6),(8) using meshes $m = 1,2,3$, and $\hat{\gamma}_4$ that for meshes $m = 2,3,4$. Then

  $$|\hat{\gamma}_3 - \hat{\gamma}_4| < 0.1(\hat{\gamma}_3 + \hat{\gamma}_4)/2 \rightarrow \text{power singularity} \quad (9)$$

• **Log Singularities**
  
  - If the stress $\sigma$ has a log singularity then

  $$\sigma = O(\ln r) \quad \text{as} \quad r \rightarrow 0 \quad (10)$$

  - When this is the case, successive increments in the FEA for $\sigma$ approach a constant value. We judge a log singularity to be present when successive increments differ by less than 10% of their mean value. That is

  $$\|\sigma_1 - \sigma_2| - |\sigma_2 - \sigma_3\| < 0.1(|\sigma_1 - \sigma_2| + |\sigma_2 - \sigma_3|)/2 \rightarrow \text{log singularity} \quad (11)$$
More Stringent (5%) Convergence-Divergence Checks

• Motivation
  – The choice of 10% in the baseline checks is merely sensible but certainly not uniquely so: By considering 5% checks instead, we can gauge to a degree how robust the performance of convergence-divergence checks of the ilk offered here is to the choice of this percentage. With this new percentage, the checks are as follows below: For these new checks, we also adopt more stringent error levels.

• Convergence vs. Divergence

\[
\begin{align*}
|\sigma_1 - \sigma_2| &> 1.05 |\sigma_2 - \sigma_3| \quad \Rightarrow \quad \text{converging} \quad (1') \\
|\sigma_1 - \sigma_2| &\leq 1.05 |\sigma_2 - \sigma_3| \quad \Rightarrow \quad \text{divergence} \quad (2')
\end{align*}
\]

• Case 1: Converged Stresses

\[
\begin{align*}
1 \leq \hat{\varepsilon} \text{ (%) } < 5 \quad &\Rightarrow \quad \text{satisfactory} \\
0.1 \leq \hat{\varepsilon} \text{ (%) } < 1 \quad &\Rightarrow \quad \text{good} \\
\hat{\varepsilon} \text{ (%) } < 0.1 \quad &\Rightarrow \quad \text{excellent}
\end{align*}
\]

where \( \hat{\varepsilon} \) continues to be as in (4),(5),(6)

• Case 2: Stresses Exhibiting Divergence

\[
|\hat{\gamma}_3 - \hat{\gamma}_4| < 0.05(\hat{\gamma}_3 + \hat{\gamma}_4)/2 \quad \Rightarrow \quad \text{power singularity} \quad (9')
\]

\[
\begin{align*}
&\left|\sigma_1 - \sigma_2\right| - \left|\sigma_2 - \sigma_3\right| < 0.05\left|\sigma_1 - \sigma_2\right| + \left|\sigma_2 - \sigma_3\right|/2 \quad \text{and} \\
&\left|\sigma_2 - \sigma_3\right| - \left|\sigma_3 - \sigma_4\right| < 0.05\left|\sigma_2 - \sigma_3\right| + \left|\sigma_3 - \sigma_4\right|/2 \quad \Rightarrow \quad \text{log singularity} \quad (11')
\end{align*}
\]

Otherwise simply not yet converged.
An Example of the Application of the Convergence-Divergence Checks

• **Singular Trial Problem**
  - Plate fixed on one edge \( (x = 0) \) and subjected to a uniform pressure \( p \) on its upper edge \( (y=W) \). For Poisson’s ratio equal to 0.069, there is a power singularity in the upper corner (Williams, 1952) with

\[
\sigma_x = O(r^{-0.1}) \text{ as } r \to 0
\]

• **FEA**
  - Uniform meshes of four node quadrilateral elements (PLANE42). Initial mesh \( (m=1) \) has 8 elements (indicated above). Mesh refinement by halving element sides. Thus \( m=2 \) has 32 elements, \( m=3 \), 128, and so on.
Example Cont’d

• Results

<table>
<thead>
<tr>
<th>m</th>
<th>$\bar{\sigma}_x$</th>
<th>$\Delta \bar{\sigma}_x$</th>
<th>CD(10%)</th>
<th>CD(5%)</th>
<th>$\hat{\epsilon}$</th>
<th>$\hat{\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.4516</td>
<td>2.439</td>
<td>d</td>
<td>c</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11.891</td>
<td>2.219</td>
<td>d</td>
<td>d</td>
<td>61</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>14.110</td>
<td>2.161</td>
<td>d</td>
<td>d</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16.271</td>
<td>2.218</td>
<td>d</td>
<td>d</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>18.489</td>
<td>2.331</td>
<td>d</td>
<td>d</td>
<td>0.038</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>20.820</td>
<td>2.478</td>
<td>d</td>
<td>d</td>
<td>0.072</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>23.298</td>
<td>2.647</td>
<td>d</td>
<td>d</td>
<td>0.088</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>25.945</td>
<td>2.832</td>
<td>d</td>
<td>d</td>
<td>0.095</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>28.777</td>
<td></td>
<td>d</td>
<td>d</td>
<td>0.097</td>
<td></td>
</tr>
</tbody>
</table>

$\bar{\sigma}_x = \sigma_x(x = 0, y = W) / p$

$\Delta \bar{\sigma}_x = \bar{\sigma}_x$ (for $m = 2$) $- \bar{\sigma}_x$ (for $m = 1$),...

CD ... convergence-divergence checks (10% or 5%)

c ... converging
d ... divergence

• Comments

– With the 10% checks, divergence is appropriately predicted by (2) throughout the mesh sequence. Once increments actually increase, (6),(8) can be used to obtain $\hat{\gamma}$. This approaches the true value of 0.1, and on meshes 6,7,8 a power singularity is correctly identified.

– With the 5% checks, converging is incorrectly predicted by (1’) on $m = 1,2,3$. This is because converging regular contributions in this example hide the weak power singularity present. However, $\hat{\epsilon}$ of (4) on this mesh sequence is unacceptable (cf (3’)), so $\bar{\sigma}_x$ is rejected ($\hat{\epsilon}$ is increased by about a factor of 4 here by $\Gamma$ of (5),(6) because of the slow “convergence”). Thereafter on $m = 2,3,4$, divergence is appropriately predicted, and ultimately on meshes 7,8,9 a power singularity identified.
Validation of Convergence-Divergence Checks

• Diverging Trial Problems
  – The 10% and 5% checks have been applied to a number of problems with known stress singularities in Sinclair et al., 2006, and Sinclair and Beisheim, 2008. Performance is summarized in the table below.

<table>
<thead>
<tr>
<th>Type of singularity</th>
<th>No. of problems</th>
<th>No. of numerical experiments</th>
<th>No. of FEA stresses rejected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power as in (7)</td>
<td>20</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>Log as in (10)</td>
<td>14</td>
<td>51</td>
<td>51</td>
</tr>
</tbody>
</table>

  – This uniform rejection of singular stresses holds true irrespective of whether the 10% or the 5% checks are used. Thus both are successful in this all-important aspect. Actual identification of a singularity being present typically requires more mesh refinement with the 5% checks (see references for details).

• Converging Test Problems
  – The 10% and 5% checks have been applied to 24 test problems with known analytical solutions in ibid, and 152 numerical experiments performed. The 10% checks, when able to be applied, provided accurate error estimates for 122 of these experiments, and conservative estimates for 23. The 5% checks provided accurate estimates for 136 and conservative estimates for 14 (see references for details). Thus the 5% checks performed slightly better than the 10% re error estimation.
Concluding Remarks

• In experiments to date, both the baseline (10%) convergence-divergence checks and the more stringent (5%) convergence-divergence checks never accept the necessarily finite FEA values that result for stresses that are actually singular and so actually have infinite values.

• The baseline checks typically require less computation than the more stringent checks to identify the presence of a stress singularity. Thus the baseline checks typically halt fruitless FEA more quickly.

• The more stringent checks typically estimate errors in non-singular problems a little better than the baseline checks.
References

• Sinclair, G.B., and Beisheim, J.R., 2008, Simple convergence-divergence checks for finite element stresses, Report ME-MSI-08, Department of Mechanical Engineering, Louisiana State University, Baton Rouge, LA (copies available here).