Eigenvalues of Trusses and Beams Using the Accurate Element Method

Maty Blumenfeld
Department of Strength of Materials
Universitatea Politehnica Bucharest, Romania
Paul Cizmas
Department of Aerospace Engineering
Texas A&M University
College Station, Texas

Abstract

The accurate element method (AEM) is a method developed for the numerical integration of the ordinary differential equations. The differential equations are discretized by dividing the computational domain in elements and reducing the solution to nodal values, similar to the finite element method. The solution over the elements is approximated using high-degree interpolation functions. A high-degree interpolation function would require a large number of unknowns per element. A prominent attribute of the AEM is the methodology developed for eliminating unknowns inside the element. As a result, the salient feature of the AEM is the decoupling between the solution accuracy and the number of unknowns. Consequently, high accuracy solutions are obtained using a reduced computational cost compared to traditional methods, such as the finite element method. The AEM uses the same approach to solve initial value problems, boundary value problems or eigenvalue problems. This paper is focused on computing the eigenvalues and eigenvectors for the axial vibration of trusses and transverse vibration of beams. The results obtained using the AEM were compared against finite element results obtained using ANSYS. For both trusses and beams, the accuracy of the eigenvalues computed using the AEM was several orders of magnitude higher than that of the finite element analysis, while the computational time was approximately the same.

Introduction

The various numerical methods recently developed in engineering may be considered as serving a common goal: solving in the best possible way the ordinary or partial differential equations describing the physical phenomena. This paper, restricted at this initial stage to solving ordinary differential equations (ODEs), presents a new method that allows generating accurate results with a small computational effort. This method, which we call the accurate element method (AEM), will be used herein to compute the eigenvalues of straight trusses and beams with constant and stepwise variable cross-section.

The AEM can be compared with the displacement method (DM) and the finite element method (FEM). The strategy of these methods can be summarized by the following three steps: (1) discretization of the domain D on which the governing equations must be integrated; (2) local approximation of the solution of the governing equations, in which the information given by each element is transferred to the nodes of the element; and (3) reconstruction of the domain D, obtained by writing nodal equations that bring together the information given by the elements adjacent to each node. The solution of the system of equations obtained in this way represents the nodal unknowns.

The first and third steps of the AEM and the DM/FEM are similar. Differences between the AEM and the DM/FEM exist in the second step. The second step is crucial in obtaining an accurate solution of the governing equations. As will be shown herein, the solution accuracy obtained with the AEM using one or a few elements can be reached by the DM only if a large number of elements is used.

Two important concepts introduced by the AEM will be presented in the next section: the complete transfer relation (CTR) and the concordant functions (CF). These two concepts will then be applied to calculate eigenvalues and eigenvectors for the axial vibration of trusses. The AEM results will be compared against the DM results. The section preceding the conclusions will present the computation of eigenvalues for the transverse vibration of straight beams.

The Accurate Element Method

The AEM has been developed and implemented to integrate linear and nonlinear ODEs with constant and variable coefficients [1]. An important aspect of the AEM is the complete transfer relation. To introduce the complete transfer relation, let us consider the ODE that describes the beam deflection [5], p. 222

$$\frac{d^4 w}{dx^4} = \frac{p}{EI}. \tag{1}$$

This equation links the fourth derivative of the displacement to the transverse distributed load p, the elasticity modulus E and the moment of inertia I. To accurately solve the ODE it is necessary to integrate it exactly four times. The ODE integration is usually done numerically, either using Runge-Kutta or similar type methods [2], or by accepting from the beginning an approximation, such as in the DM where the variation of the distributed load is neglected [1]. Instead, the AEM integrates the ODE without any approximation, leading to one or more integral equations. For the solution of equation (1), the AEM generates four integral equations, each one including a term

$$\int_{x1}^{x2} f(x)\,\phi(x)\,dx, \tag{2}$$

where $f(x)$ is a known function and x_1, x_2 are the limits of the integration domain D. The ensemble of the four equations previously mentioned will be referred as the complete transfer relation (CTR), because it transfers the information to the nodes, and because this transfer is complete, *i.e.*, without any approximation. The problem is not yet solved because equation (2) cannot be integrated since ϕ(x) is unknown. In order to make the integration possible, ϕ(x) is usually replaced by an interpolation function, which is a low-degree polynomial leading therefore to approximate results. Instead, the AEM uses high-degree interpolation functions, called herein concordant functions. To eliminate part of the unknowns, the governing equation and its derivatives are used, as showed in the next section.

The eigenvalues of trusses

The governing equation

The behavior of a truss is described by two differential equations: (1) the equilibrium equation, and (2) the deformation equation. The equilibrium equation is

$$q(x) = -\frac{dN}{dx}, \tag{3}$$

where $q(x)$ is the distributed axial load and N is the internal axial force. The deformation equation, based on Bernoulli hypothesis, is

$$\frac{du}{dx} = \frac{N}{EA}, \tag{4}$$

where the axial displacement is $u = u(x)$ and A is the constant transverse area of the truss. If equation (3) is substituted into the differentiated form of equation (4), the following second-order ODE is obtained

$$\frac{d^2 u}{dx^2} + \frac{q(x)}{EA} = 0. \tag{5}$$

The eigenvalue problem results by replacing $q(x)$ with an inertia load. If ρ is the density of the material and ω is the circular frequency, the inertia force dF is given by the product between the mass ($dm = \rho A dx$) and the acceleration ($u\omega^2$). As a result, $dF = u\omega^2 \rho A\, dx$ and therefore

$$q(x) = dF/dx = u\omega^2 \rho A. \tag{6}$$

If $q(x)$ is substituted in (5) and $u(x)$ is replaced by ϕ (x), then $\frac{d^2\phi}{dx^2} + \beta\,\phi(x) = 0$ or

$$\phi^{(2)} + \beta\phi^{(0)} = 0, \tag{7}$$

where $\beta = (\rho/E)\,\omega^2$ and $\phi^{(k)} = \frac{d^k\phi}{dx^k}$.

The numerical examples analyzed below will calculate the eigenvalues of equation (7), with the following boundary conditions, corresponding to $x_1 = 0$ and $x_2 = 1$

$$\phi(x = x_1 = 0) = \phi_1^{(0)} = 0 \tag{8}$$

$$\phi'(x = x_2 = 1) = \phi_2^{(1)} = 0. \tag{9}$$

The exact analytic solution of (7) is $\phi(x) = A\sin\sqrt{\beta}\,x + B\cos\sqrt{\beta}\,x$. The boundary condition (8) yields $B = 0$. The boundary condition (9), applied to the derivative $\phi'(x) = A\sqrt{\beta}\cos\sqrt{\beta}\,x$, gives $\cos(\sqrt{\beta}) = 0$. Consequently $\sqrt{\beta} = (2k-1)\frac{\pi}{2}$, $\quad k = 1, 2, 3, \ldots$ and

$$\beta_k = \{\, (2k-1)\pi/2 \,\}^2,\ k = 1, 2, 3, \ldots \tag{10}$$

The Complete Transfer Relation

Let us suppose that the integration of equation (7) must be performed between x_1 and x_2 , the current abscissa being x. After the first integration, equation (7) becomes

$$\phi^{(1)} + \beta\int_{x1}^{x}\phi^{(0)}(x)\,dx + K_1 = 0, \tag{11}$$

where K_1 is an integration constant. The integration constant K_1 can be eliminated by evaluating equation (11) at both ends of the integration interval. At $x = x_1$, it results

$$K_1 = -\phi_1^{(1)} \tag{12}$$

and at $x = x_2$

$$K_1 = -\phi_2^{(1)} - \beta\int_{x1}^{x2}\phi^{(0)}(x)dx. \tag{13}$$

If K_1 is eliminated between equations (12) and (13), the first integral of equation (7) is obtained

$$\phi_1^{(1)} - \phi_2^{(1)} - \beta\int_{x1}^{x2}\phi^{(0)}(x)dx = 0. \tag{14}$$

The procedure continues with the integration of equation (11)

$$\phi^{(0)} + \beta\int_{x1}^{x}\left(\int_{x1}^{x}\phi^{(0)}(\xi)\,d\xi\right)dx + K_1 x + K_2 = 0 \tag{15}$$

To eliminate the integration constant K_2, equation (15) is evaluated at both ends of the integration interval. At $x = x_1$, one obtains

$$\phi_1^{(0)} + \beta \int_{x1}^{x1} \left(\int_{x1}^{x} \phi^{(0)}(\xi)\, d\xi \right) dx + K_1 x_1 + K_2 = 0, \tag{16}$$

and at $x = x_2$

$$\phi_2^{(0)} + \beta \int_{x1}^{x2} \left(\int_{x1}^{x} \phi^{(0)}(\xi)\, d\xi \right) dx + K_1 x_2 + K_2 = 0 \tag{17}$$

The integration constant K_1 is replaced in (16) by (12) and in (17) by (13). Taking into account that $\int_{x1}^{x1} \left(\int_{x1}^{x} \phi^{(0)}(\xi)\, d\xi \right) dx = 0$, the difference between (16) and (17) leads to

$$\phi_1^{(0)} - \phi_2^{(0)} - \phi_1^{(1)} x_1 + \phi_2^{(1)} x_2 + \beta \left\{ -\int_{x1}^{x2} \left(\int_{x1}^{x} \phi^{(0)}(\xi)\, d\xi \right) dx + x_2 \int_{x1}^{x2} \phi^{(0)} dx \right\} = 0 \tag{18}$$

Using integration by parts, the terms from the parenthesis in equation (18) reduce to

$$-\int_{x1}^{x2} \left(\int_{x1}^{x} \phi^{(0)}(\xi)\, d\xi \right) dx + x_2 \int_{x1}^{x2} \phi^{(0)} dx = \int_{x1}^{x2} x\, \phi^{(0)} dx \tag{19}$$

Consequently, equation (7) integrated twice becomes

$$\phi_1^{(0)} - \phi_2^{(0)} - \phi_1^{(1)} x_1 + \phi_2^{(1)} x_2 + \beta \int_{x1}^{x2} x\, \phi^{(0)}(x) dx = 0. \tag{20}$$

The ensemble of the two relations (14) and (20), which is the result of transferring the two successive integrals of (7) to the ends of the computational domain, represents the complete transfer relations (CTR).

The Concordant Functions

The third-degree concordant function CF4

As it was stated previously the challenge is how to replace function $\phi(x)$ in order to integrate it. The function that approximates $\phi(x)$ must be based on the four end unknowns which have been used in the in equations (14) and (20), namely $\phi_1^{(0)}, \phi_2^{(0)}, \phi_1^{(1)}$ and $\phi_2^{(1)}$. Based on these unknowns, one obtains a unique description of a third-degree polynomial

$$\phi^{(0)}(x) = C_0 + C_1\, x + C_2\, x^2 + C_3\, x^3 \tag{21}$$

whose first derivative is

$$\phi^{(1)}(x) = d\phi^{(0)}/dx = C_1 + 2\, C_2\, x + 3\, C_3\, x^2 \tag{22}$$

The relation (21) will be further referred to as the CF4 concordant (or interpolation) function. Its coefficients can be obtained by imposing the following end conditions:

$$\phi^{(0)}(x = x_1) = C_0 + C_1\, x_1 + C_2\, x_1^2 + C_3\, x_1^3 = \phi_1^{(0)} \tag{23}$$

$$\phi^{(0)}(x = x_2) = C_0 + C_1\, x_2 + C_2\, x_2^2 + C_3\, x_2^3 = \phi_2^{(0)} \tag{24}$$

$$\phi^{(1)}(x = x_1) = C_1 + 2\, C_2\, x_1 + 3\, C_3\, x_1^2 = \phi_1^{(1)} \tag{25}$$

$$\phi^{(1)}(x=x_2)=C_1+2C_2\,x_2+3C_3\,x_2^2=\phi_2^{(1)} \tag{26}$$

These four conditions yield the system of equations

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2\,x_1 & 3x_1^2 \\ 0 & 1 & 2\,x_2 & 3x_2^2 \end{bmatrix}\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}=\begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix}$$

or using the notation

$$[A]_4\left[\overline{C}\right]_4=\left[\overline{\phi}\right]_4 \tag{27}$$

where $[A]_4$ is the square matrix and

$$\left[\overline{C}\right]_4=\left[C_0 \quad C_1 \quad C_2 \quad C_3\right]^T \tag{28}$$

$$\left[\overline{\phi}\right]_4=\left[\phi_1^{(0)} \quad \phi_2^{(0)} \quad \phi_1^{(1)} \quad \phi_2^{(1)}\right]^T. \tag{29}$$

The vector of coefficients $[\overline{C}]_4$ can be obtained using the inverse of the matrix $[A]_4$

$$\left[\overline{C}\right]_4=[A]_4^{-1}\left[\overline{\phi}\right]_4. \tag{30}$$

The integrals remaining in equations (14) and (20) can now be evaluated by using equation (21)

$$Int\phi 4=\int_{x1}^{x2}\phi^{(0)}dx=\int_{x1}^{x2}\left(C_0+C_1\,x+C_2\,x^2+C_3\,x^3\right)dx=\left(x_2-x_1\right)C_0+\frac{x_2^2-x_1^2}{2}C_1+\frac{x_2^3-x_1^3}{3}C_2+\frac{x_2^4-x_1^4}{4}C_3$$

$$Int\,x\phi 4=\int_{x1}^{x2}x\phi^{(0)}dx=\int_{x1}^{x2}\left(C_0x+C_1\,x^2+C_2\,x^3+C_3\,x^4\right)dx=\frac{x_2^2-x_1^2}{2}C_0+\frac{x_2^3-x_1^3}{3}C_1+\frac{x_2^4-x_1^4}{4}C_2+\frac{x_2^5-x_1^5}{5}C_3$$

If one denotes

$$\left[Int\right]_4=\left[x_2-x_1 \quad \frac{x_2^2-x_1^2}{2} \quad \frac{x_2^3-x_1^3}{3} \quad \frac{x_2^4-x_1^4}{4}\right] \tag{31}$$

$$\left[Int\,x\right]_4=\left[\frac{x_2^2-x_1^2}{2} \quad \frac{x_2^3-x_1^3}{3} \quad \frac{x_2^4-x_1^4}{4} \quad \frac{x_2^5-x_1^5}{5}\right] \tag{32}$$

then

$$Int\phi 4=\left[Int\right]_4\left[\overline{C}\right]_4=\left[Int\right]_4\left[A\right]_4^{-1}\left[\overline{\phi}\right]_4=\left[\overline{P}\right]_4\left[\overline{\phi}\right]_4=P_1\phi_1^{(0)}+P_2\phi_2^{(0)}+P_3\phi_1^{(1)}+P_4\phi_2^{(1)} \tag{33}$$

$$Int\,x\phi 4=\left[Intx\right]_4\left[\overline{C}\right]_4=\left[Intx\right]_4\left[A\right]_4^{-1}\left[\overline{\phi}\right]_4=\left[\overline{Q}\right]_4\left[\overline{\phi}\right]_4=Q_1\phi_1^{(0)}+Q_2\phi_2^{(0)}+Q_3\phi_1^{(1)}+Q_4\phi_2^{(1)} \tag{34}$$

where $\left[\bar{P}\right]_4$ and $\left[\bar{Q}\right]_4$ are two matrices with four terms, which can be computed for each pair of abscissas x_1 and x_2. If x_1=0 and x_1=1, the terms of $\left[\bar{P}\right]_4$ and $\left[\bar{Q}\right]_4$ are those given in the CF4 rows of the Tables 1 and 2, respectively.

It is now possible to obtain a final form the CTR by replacing (33) and (34) in both equations (14) and (20)

$$\phi_1^{(1)} - \phi_2^{(1)} - \beta\left(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)}\right) = 0 \tag{35}$$

$$-\phi_1^{(0)} + \phi_2^{(0)} + x_1\phi_1^{(1)} - x_2\phi_2^{(1)} - \beta\left(Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)}\right) = 0 \tag{36}$$

Table 1. The coefficients P_i used in equation (35) for different CFs

	Coefficients P							
	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
CF4	1/2	1/2	1/12	-1/12	0	0	0	0
CF6	1/2	1/2	1/10	-1/10	1/120	1/120	0	0
CF8	1/2	1/2	3/28	-3/28	1/84	1/84	1/1680	-1/1680

Table 2. The coefficients Q_i used in equation (36) for different CFs

	Coefficients Q							
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
CF4	3/20	7/20	1/30	-1/20	0	0	0	0
CF6	1/7	5/14	4/105	-13/120	1/280	1/210	0	0
CF8	5/36	13/36	5/126	-17/252	5/1008	1/144	1/3780	-1/3024

The CTR includes two equations, (35) and (36), which have four unknowns. The problem can be solved, however, because the boundary conditions (8) and (9) are also available. The four equations thus obtained form a homogeneous system that can be written as

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 1 & x_1(=0) & -x_2(=-1) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + \beta\begin{bmatrix} -P_1 & -P_2 & -P_3 & -P_4 \\ -Q_1 & -Q_2 & -Q_3 & -Q_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

In the system of equations (37), the first row is equation (35), the second row is equation (36), the third row is equation (8) and the last row is equation (9). The system of equations can be written in compact form as

$$\Big[\ [A_0] + \beta[A_1]\ \Big][\bar{\phi}]_4 = [0]. \tag{38}$$

Equation 38 represents a classical eigenvalue problem. The eigenvalues corresponding to the CF4 interpolation functions, obtained by solving equation (37), are shown in Table 3. Table 3 also shows the exact solution (10) and the relative errors

$$error = (eigenvalue_{CF4} - eigenvalue_{exact})/eigenvalue_{exact} \tag{39}$$

computed with respect to the exact solution (10).

Table 3. The exact and the AEM solutions based on low-degree polynomials

	Errors (referred to the exact analytic solution)					
NE	IF2			CF4		
⇓	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
1	*	*	*	1 E-1	*	*
2	1 E-1	*	*	3.7 E-2	1 E-1	2 E-1
3	4.6 E-2	1.6E-1	*	7.9 E-4	1 E-2	1 E-1
4	2.6 E-2	1 E-1	1.9 E-1	2.6 E-4	3.7 E-3	1.6 E-2
5	1.6 E-2	6 E-2	1.4 E-1	1 E-4	1.6 E-3	7.4 E-3
10	4 E-3	1.6 E-2	3.7 E-2	6.7 E-6	1.1 E-4	5.2 E-4
20	1 E-3	4 E-3	9 E-3	4.2 E-7	6.7 E-6	3.4 E-5
30	4.6 E-4	1.8 E-3	4.1 E-3	8.3 E-8	1.3 E-6	6.7 E-6
40	2.6 E-4	1 E-3	2.3 E-3	*	*	*
50	1.6 E-4	6.6 E-4	1.5 E-3	*	*	*

Note that equation (21) represents a Hermite type polynomial. The methodology presented above will be used in the next sections to obtain higher-degree CFs. The relative error (39) allows to evaluate how many digits of the computed result coincide with exact solution. The result for CF8 corresponding to the first eigenfrequency has an error of 1.4 E-6. The exponent of the error, 6 in this case, indicates a coincidence of 6 digits. This is not always true, but a coincidence of 6 ± 1 digits is to be expected for the CF8 interpolation function.

The fifth-degree concordant function CF6

Let us suppose that the Concordant Function is a fifth-degree polynomial

$$\phi^{(0)}(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 \tag{40}$$

where C_0, C_1, C_2, C_3, C_4 and C_5 are six unknown constants. These six constants will be obtained by using six conditions. Four of these conditions are identical to equations (25-28) presented in the previous section. The other two necessary conditions will be obtained by using the second derivative $\phi^{(2)}$ at the ends of the domain:

$$\phi^{(2)}(x = x_1) = \phi_1^{(2)} \tag{41}$$

$$\phi^{(2)}(x = x_2) = \phi_2^{(2)} \tag{42}$$

Based on these six conditions a system of equations similar to (27) can be written

$$[A]_6 [\overline{C}]_6 = [\overline{\phi}]_6, \tag{43}$$

where $[A]_6$ is a six-by-six square matrix and

$$[\overline{C}]_6 = \begin{bmatrix} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \end{bmatrix}^T$$

$$\left[\overline{\phi}\right]_6 = \left[\phi_1^{(0)} \quad \phi_2^{(0)} \quad \phi_1^{(1)} \quad \phi_2^{(1)} \quad \phi_1^{(2)} \quad \phi_2^{(2)}\right]^T. \tag{44}$$

The vector of coefficients $\left[\overline{C}\right]_6$ can be obtained using the inverse of the $\left[A\right]_6$ matrix

$$\left[\overline{C}\right]_6 = \left[A\right]_6^{-1}\left[\overline{\phi}\right]_6. \tag{45}$$

The integrals of equations (14) and (20) will be denoted as *Int* ϕ *6* and *Int x* ϕ *6*. These integrals can be evaluated similarly to the integrals *Int* ϕ *4* and *Int x* ϕ *4*, if equation (40) is used instead of equation (21). If one denotes

$$\left[Int\right]_6 = \left[x_2 - x_1 \quad \frac{x_2^2 - x_1^2}{2} \quad \frac{x_2^3 - x_1^3}{3} \quad \frac{x_2^4 - x_1^4}{4} \quad \frac{x_2^5 - x_1^5}{5} \quad \frac{x_2^6 - x_1^6}{6}\right]$$

$$\left[Int\ x\right]_6 = \left[\frac{x_2^2 - x_1^2}{2} \quad \frac{x_2^3 - x_1^3}{3} \quad \frac{x_2^4 - x_1^4}{4} \quad \frac{x_2^5 - x_1^5}{5} \quad \frac{x_2^6 - x_1^6}{6} \quad \frac{x_2^7 - x_1^7}{7}\right]$$

then

$$Int\phi 6 = \left[Int\right]_6\left[\overline{C}\right]_6 = \left[Int\right]_6\left[A\right]_6^{-1}\left[\overline{\phi}\right]_6 = \left[\overline{P}\right]_6\left[\overline{\phi}\right]_6 = P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)} + P_5\phi_1^{(2)} + P_6\phi_2^{(2)} \tag{46}$$

$$Intx\phi 6 = \left[Intx\right]_6\left[\overline{C}\right]_6 = \left[Intx\right]_6\left[A\right]_6^{-1}\left[\overline{\phi}\right]_6 = \left[\overline{Q}\right]_4\left[\overline{\phi}\right]_4 = Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)} + Q_5\phi_1^{(2)} + Q_6\phi_2^{(2)} \tag{47}$$

where $\left[\overline{P}\right]_6$ and $\left[\overline{Q}\right]_6$ are one-row matrices with six terms, which can be computed for each pair of abscissas x_1 and x_2. If, for instance, x_1=0 and x_2=1, the terms of $\left[\overline{P}\right]_6$ and $\left[\overline{Q}\right]_6$ are those given in the CF6 rows of the Tables 1 and 2, respectively.

It is now possible to obtain the final form of the CTR by replacing equations (46) and (47) in equations (14) and (20)

$$\begin{aligned} &\phi_1^{(1)} - \phi_2^{(1)} - \beta\left(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)} + P_5\phi_1^{(2)} + P_6\phi_2^{(2)}\right) = 0 \\ -&\phi_1^{(0)} + \phi_2^{(0)} + x_1\phi_1^{(1)} - x_2\phi_2^{(1)} - \beta\left(Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)} + Q_5\phi_1^{(2)} + Q_6\phi_2^{(2)}\right) = 0 \end{aligned} \tag{48}$$

A new problem arises now, whose solution is the essence of the accurate element method presented herein. The conditions of the problem have not changed, therefore only four equations are available, while there are six unknowns involved, corresponding to (44). The unknowns $\phi_1^{(0)}, \phi_2^{(0)}, \phi_1^{(1)}$ and $\phi_2^{(1)}$ were used to determine the CF4 interpolation function. We will consider these unknowns to be the basic unknowns, while the unknowns $\phi_1^{(2)}$ and $\phi_2^{(2)}$ will be considered the apparent unknowns, and they will be eliminated.

To eliminate the apparent unknowns $\phi_1^{(2)}$ and $\phi_2^{(2)}$ additional information is needed. Such information is usually obtained from some relations with adjacent elements or by accepting a reasonable approximation. Neither of these options is used by the AEM. Instead, the necessary information is obtained from the governing equations. The second derivative is obtained from the governing equation (7), that is

$$\phi^{(2)} = -\beta\,\phi^{(0)}. \tag{49}$$

As a result, the second derivatives at the ends of the integration domain are given by two exact additional relations

$$\phi_1^{(2)} = -\beta\phi_1^{(0)} \tag{50}$$

$$\phi_2^{(2)} = -\beta\phi_2^{(0)} \tag{51}$$

If the derivatives $\phi_1^{(2)}, \phi_2^{(2)}$ are replaced in equation (48) it results

$$\begin{cases} \phi_1^{(1)} - \phi_2^{(1)} - \beta\left(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)}\right) + \beta^2\left(P_5\phi_1^{(0)} + P_6\phi_2^{(0)}\right) = 0 \\ -\phi_1^{(0)} + \phi_2^{(0)} + x_1\phi_1^{(1)} - x_2\phi_2^{(1)} - \beta\left(Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)}\right) + \beta^2\left(Q_5\phi_1^{(0)} + Q_6\phi_2^{(0)}\right) = 0 \end{cases} \tag{52}$$

Equations (50), (51) and (52) can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & 1 & x_1 & -x_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + \beta\begin{bmatrix} -P_1 & -P_2 & -P_3 & -P_4 \\ -Q_1 & -Q_2 & -Q_3 & -Q_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} + \beta^2\begin{bmatrix} P_5 & P_6 & (P_7) & (P_8) \\ Q_5 & Q_6 & (Q_7) & (Q_8) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1^{(0)} \\ \phi_2^{(0)} \\ \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{53}$$

Note that equation (53) is identical to equation (37) except for the additional term that includes β^2. For the interpolation functions CF6, the terms P_7, P_8, Q_7 and Q_8 are null. The system of equations (53) can be written in compact form as

$$\left[\left[A_0\right] + \beta\left[A_1\right] + \beta^2\left[A_2\right]\right]\left[\overline{\phi}\right]_4 = \left[0\right]. \tag{54}$$

The equation (54) represents a polynomial eigenvalue problem. The eigenvalues β were obtained herein using the routine POLYEIG of MATLAB. This function solves the polynomial eigenvalue problem of degree p. If either one but not both [A $_0$] and [A $_p$] are singular, the problem is well posed, but some of the eigenvalues may be zero or infinite. From equation (54) three eigenvalues result, which are given in the CF6 row of Table 3, together with the relative errors. Note that the first four digits of the first eigenvalue are exact although only one AEM element with CF6 interpolation functions has been used.

The seventh-degree concordant function CF8

As shown in the previous section, the results improve if the degree of the CF increases. The methodology presented for the elimination of the apparent unknowns can be applied for any higher-degree CF. For instance if a CF8 with eight terms is considered

$$\phi^{(0)}(x) = C_0 + C_1\,x + C_2\,x^2 + C_3\,x^3 + C_4\,x^4 + C_5\,x^5 + C_6\,x^5 + C_7\,x^7 \tag{55}$$

two additional conditions compared to the CF6 case are necessary to obtain all the constants. These additional conditions will be based on the third derivative, $\phi^{(3)}$, at ends of the domain

$$\phi^{(3)}(x = x_1) = \phi_1^{(3)} \tag{56}$$

$$\phi^{(3)}(x = x_2) = \phi_2^{(3)}. \tag{57}$$

A system of equations similar to (27) can be written using the eight available conditions (25-28), (41), (42), (56) and (57)

$$[A]_8[\bar{C}]_8 = [\bar{\phi}]_8, \tag{58}$$

where $[A]_8$ is an eight-by-eight square matrix and

$$[\bar{C}]_8 = \begin{bmatrix} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{bmatrix}^T \tag{59}$$

$$[\bar{\phi}]_8 = \begin{bmatrix} \phi_1^{(0)} & \phi_2^{(0)} & \phi_1^{(1)} & \phi_2^{(1)} & \phi_1^{(2)} & \phi_2^{(2)} & \phi_1^{(3)} & \phi_2^{(3)} \end{bmatrix}^T. \tag{60}$$

The vector of coefficients $\left[\bar{C}\right]_8$ can be obtained using the inverse of the $\left[A\right]_8$

$$[\bar{C}]_8 = [A]_8^{-1}[\bar{\phi}]_8. \tag{61}$$

The integrals of equations (14) and (20) will be denoted as *Int* ϕ *8* and *Intx* ϕ *8*. These integrals can be evaluated in the same manner as the integrals *Int* ϕ *6* and *Int x* ϕ *6*, if equation (55) is used instead of equation (40). Following the same procedure as in the previous section yields

$$Int\phi 8 = P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)} + P_5\phi_1^{(2)} + P_6\phi_2^{(2)} + P_7\phi_1^{(3)} + P_8\phi_2^{(3)} \tag{62}$$

$$Intx\phi 8 = Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)} + Q_5\phi_1^{(2)} + Q_6\phi_2^{(2)} + Q_7\phi_1^{(3)} + Q_8\phi_2^{(3)}, \tag{63}$$

where the eight terms P_i, Q_i, are given for x_1=0 and x_2=1 in the CF8 rows of the Tables 1 and 2.

The final form of the CTR results by replacing (62) and (63) in equations (14) and (20)

$$\phi_1^{(1)} - \phi_2^{(1)} - \beta\left(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)} + P_5\phi_1^{(2)} + P_6\phi_2^{(2)} + P_7\phi_1^{(3)} + P_8\phi_2^{(3)}\right) = 0 \tag{64}$$

$$-\phi_1^{(0)} + \phi_2^{(0)} + x_1\phi_1^{(1)} - x_2\phi_2^{(1)} - \beta\left(Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)} + Q_5\phi_1^{(2)} + Q_6\phi_2^{(2)} + Q_7\phi_1^{(3)} + Q_8\phi_2^{(3)}\right) = 0 \tag{65}$$

Because the basic unknowns remain $\phi_1^{(0)}, \phi_2^{(0)}, \phi_1^{(1)}$ and $\phi_2^{(1)}$, the number of apparent unknowns is now four, namely $\phi_1^{(2)}, \phi_2^{(2)}, \phi_1^{(3)}$ and $\phi_1^{(3)}$. The apparent unknowns must be eliminated using the governing equation. Besides the two relations (50) and (51) established previously, two more relations can be obtained by differentiating equation (49)

$$\phi^{(3)} = -\beta\,\phi^{(1)}. \tag{66}$$

As a result, two exact additional relations specify the third derivatives at the ends of the integration domain

$$\phi_1^{(3)} = -\beta\phi_1^{(1)} \tag{67}$$

$$\phi_2^{(3)} = -\beta\phi_2^{(1)}. \tag{68}$$

If equations (50), (51), (67) and (68) are replaced in equation (64), this yields

$$\begin{cases} \phi_1^{(1)} - \phi_2^{(1)} - \beta\left(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)}\right) + \beta^2\left(P_5\phi_1^{(0)} + P_6\phi_2^{(0)} + P_7\phi_1^{(1)} + P_8\phi_2^{(2)}\right) = 0 \\ -\phi_1^{(0)} + \phi_2^{(0)} + x_1\phi_1^{(1)} - x_2\phi_2^{(1)} - \beta\left(Q_1\phi_1^{(0)} + Q_2\phi_2^{(0)} + Q_3\phi_1^{(1)} + Q_4\phi_2^{(1)}\right) + \beta^2\left(Q_5\phi_1^{(0)} + Q_6\phi_2^{(0)} + Q_7\phi_1^{(1)} + Q_8\phi_2^{(2)}\right) = 0. \end{cases}$$

The final system of equations in matrix form is given by equation (53). Consequently, equations (54) remain the same, leading to the four eigenvalues that are given in the CF8 row of Table 3. Note that the first six digits of the first eigenvalue are exact although only one AEM element with CF8 interpolation functions has been used.

A numerical comparison between the DM and the AEM

The DM solution is based on the two-by-two stiffness matrix $[K]$, leading to the well-known relation [4], p. 11

$$[\overline{F}] = [K][\overline{\delta}], \tag{69}$$

where $[\overline{F}] = [F_{x1} \quad F_{x2}]^T$ is the nodal forces vector and $[\overline{\delta}] = [u_1 \quad u_2]^T$ is the nodal axial displacements vector. Because $[\overline{\delta}]$ is based on only two end unknowns, the interpolation function used to replace $u(x) = \phi(x)$ has only a linear variation given by

$$u(x) = u_1 + (u_2 - u_1)/L = \phi_1 + (\phi_2 - \phi_1)/L \tag{70}$$

where $L = x_2 - x_1$. The interpolation function (70) is referred herein as IF2. By using IF2, the CTR becomes [1], p. 78

$$\begin{cases} \phi_1^{(0)} - \phi_2^{(0)} + L\phi_1^{(1)} = \beta\,(L^2\phi_1^{(0)}/3 + L^2\phi_2^{(0)}/6) \\ -\phi_1^{(0)} + \phi_2^{(0)} - L\phi_2^{(1)} = \beta\,(L^2\phi_1^{(0)}/6 + L^2\phi_2^{(0)}/3) \end{cases} \tag{71}$$

By applying the boundary conditions (8) and (9) and taking into account that $L = x_2 - x_1 = 1$, the second equation (71) yields $\beta_{IF2} = 3$.

The relative error with respect to the analytical value of the eigenvalue is

$$error_{IF2} = (eigenvalue_{IF2} - eigenvalue_{exact})/eigenvalue_{exact} = 0.216.$$

Note that this error is at least two-to-three orders of magnitude larger than the AEM errors shown in Table 3 for the first eigenvalues.

The rest of this section presents a comparison between the eigenvalues obtained using the IF2 and CF interpolation functions. The comparison will be done for a truss with constant area, using a different number of elements, *NE*. The truss is clamped at both ends such as the boundary conditions are

$$\phi(x = x_1 = 0) = \phi_1^{(0)} = u_1 = 0 \tag{72}$$

$$\phi(x = x_2 = \pi) = \phi_2^{(0)} = u_2 = 0. \tag{73}$$

The length of the truss is $L = \pi$. If $\rho/E = 1$, the governing equation (7) can be written as

$$\frac{d^2\phi}{dx^2} + \omega^2\,\phi(x) = 0. \tag{74}$$

The exact eigenfrequencies of equation (74) with the boundary conditions (72) and (73) are given by ω_1 =1, ω_2 =2, ω_3 =3, ω_1 =4,…, ω_k =k.

Table 4. Errors for the IF2 and CF4 interpolation functions

NE ⇓	Errors (referred to the exact analytic solution)					
	IF2			CF4		
	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
1	*	*	*	1 E-1	*	*
2	1 E-1	*	*	3.7 E-2	1 E-1	2 E-1
3	4.6 E-2	1.6E-1	*	7.9 E-4	1 E-2	1 E-1
4	2.6 E-2	1 E-1	1.9 E-1	2.6 E-4	3.7 E-3	1.6 E-2
5	1.6 E-2	6 E-2	1.4 E-1	1 E-4	1.6 E-3	7.4 E-3
10	4 E-3	1.6 E-2	3.7 E-2	6.7 E-6	1.1 E-4	5.2 E-4
20	1 E-3	4 E-3	9 E-3	4.2 E-7	6.7 E-6	3.4 E-5
30	4.6 E-4	1.8 E-3	4.1 E-3	8.3 E-8	1.3 E-6	6.7 E-6
40	2.6 E-4	1 E-3	2.3 E-3	*	*	*
50	1.6 E-4	6.6 E-4	1.5 E-3	*	*	*

As shown in Table 4, the results obtained using the IF2 interpolation function could be considered rather poor. Even the results obtained using 50 elements with the IF2 interpolation function are quite far from the exact analytic solution. The eigenvalues were obtained by using a program written by the authors and by using ANSYS, the results being the same. Note that commercial codes are not able to use a higher-degree interpolation function, because they are based on a stiffness matrix that allows only the use of low-degree interpolation functions.

Table 5. Errors for the CF8, CF12 and CF16 interpolation functions

ω ⇓	Errors (referred to the exact analytic solution)								
	CF8			CF12			CF16		
	NE=1	NE=2	NE=3	NE=1	NE=2	NE=3	NE=1	NE=2	NE=3
ω_1	2.8 E-4	6.8 E-7	2.8 E-8	1.3 E-8	1.9 E-11	1.4 E-13	1.7 E-11	2 E-16	8 E-16
ω_2	4 E-3	2.8 E-4	6.5 E-6	1.7 E-7	1.3 E-7	5.7 E-10	2.3 E-8	1.7 E-11	1.3 E-14
ω_3	*	2.6 E-3	2.8 E-4	1 E-2	6.9 E-6	1.3 E-7	2.1 E-4	4.7 E-9	1.7 E-11
ω_4	*	3.6 E-3	1 E-3	3.1 E-2	1.7 E-5	1.8 E-6	1.1 E-3	2.3 E-8	7.6 E-10
ω_5	*	6 E-2	5 E-3	*	1.4 E-3	2.2 E-5	5.7 E-2	9.3 E-6	2.3 E-8
ω_6	*	*	3 E-3	*	1 E-2	1.7 E-5	9.1 E-2	2.1 E-4	2.3 E-7
ω_7	*	*	4 E-2	*	2.6 E-2	7.4 E-4	*	8 E-4	3.5 E-6
ω_8	*	*	8 E-2	*	3 E-2	2.6 E-3	*	1.1 E-3	2.3 E-5
ω_9	*	*	*	*	*	1.3 E-2	*	1.3 E-2	2.1 E-4
ω_{10}	*	*	*	*	*	1.8 E-2	*	5.7 E-2	4.4 E-4

Table 5 shows the variation of the eigenvalue errors for interpolation functions CF8, CF12 and CF16. The number of elements was less or equal to three. The first ten eigenvalues, if available, were listed in the table. The eigenvalues were calculated using the routine POLYEIG of MATLAB.

Recall that the CF4 interpolation function did not require the elimination of any apparent unknowns, therefore the only difference between IF2 and CF4 is the use of a different number of unknowns per element. The result obtained for ω_1 using CF4 with three elements, that is twelve unknowns, is similar to that obtained using IF2 with thirty elements, that is sixty unknowns. Consequently, for the same accuracy, the computational time of the AEM using the CF4 interpolation function is smaller than that of the FEM using the IF2 interpolation function. One can also compare the errors between the CF8, CF12 and CF16 interpolation functions with the same number of elements. Let us consider the results corresponding to the first eigenvalue for three elements. In this case, the number of unknowns after eliminating the apparent unknowns was twelve, the same for the CF8, CF12 and CF16 interpolation functions. The ratios between the errors corresponding to the CF and the IF2 interpolation functions, shown in Table 6, indicate the net advantage of the AEM with high-degree interpolation functions.

Table 6. Errors for the CF8, CF12 and CF16 interpolation functions

$\text{Error}_{\text{IF2}}/\text{Error}_{\text{CF8}}$	$\text{Error}_{\text{IF2}}/\text{Error}_{\text{CF12}}$	$\text{Error}_{\text{IF2}}/\text{Error}_{\text{CF16}}$
1.6E4	3.3E9	5.7E11

The eigenvectors

This section briefly describes how to calculate the eigenvectors once the eigenvalues have been computed. First, the eigenvalue β is substituted in either equation (37) or (53). Because equations (37) and (53) are homogeneous systems, the first CTR equation must be replaced by an equation that gives an arbitrary nonzero value to one of the basic end unknowns (obviously not to those involved in the boundary conditions). The solution of this system of equations provides the end unknowns, $\phi_1^{(k)}$ and $\phi_2^{(k)}$. The apparent unknowns can be calculated using equations (50) and (51) for CF6 or equations (50), (51), (67) and (68) for CF8. The vector of coefficients $\left[\bar{C}\right]$ results from equations (45) or (61) and the eigenvector is obtained from (40) or (55).

If the solution obtained with the CF8 interpolation function is considered, and $\phi_2^{(0)} = 1$ is imposed arbitrarily, the eigenvector corresponding to the first eigenvalue is given by

$$\phi(x) = 0 + 1.570795735759208\,x + 0 * x^2 - 0.64596473657853\,x^3 - 0.00056694323655648\,x^4 + \\ + 0.08185097791941587\,x^5 - 0.002916594370312\,x^6 - 0.0031984394932305\,x^7.$$

The value of the eigenvector at $x = 0.5$ is given by $\phi(x = 0.5) = 0.7071041$, while the value corresponding to the exact solution (73) is 0.7071067. The first five digits of the AEM eigenvector coincide to the exact solution, although only one element was used.

Methods for improving the accuracy of the AEM

The results obtained using one element with the CF8 interpolation function have satisfactory accuracy. It is useful, however, to examine ways of improving them. Two options will be discussed herein: (1) increasing the degree of the interpolation polynomial, and (2) increasing the number of elements.

Increasing the degree of the polynomial, that is, increasing the number of terms of in the CF interpolation function can be done without special difficulties by following the procedure already described. This requires higher derivatives of the governing equation (7) in order to eliminate the new apparent unknowns. This strategy has already been used for CF16 (fifteenth-degree polynomial) and CF20 (nineteenth-degree polynomial) [1], pp. 80-81. For the case of CF20, the error corresponding to the first eigenvalue obtained with one element dropped to 8.88 x 10^{-16}, that is, the first 15 digits coincide with the exact result.

Increasing the number of elements can be done together or independently of varying the degree of the interpolation function. Let us suppose the domain of integration is divided in two elements noted $1-2$ and $3-4$, respectively. The lengths of these elements can be equal or not. The number of nodal unknowns is double the number of unknowns per element, that is, two times four. The unknowns of the element 1-2 are $\phi_1^{(0)}, \phi_2^{(0)}, \phi_1^{(1)}$ and $\phi_2^{(1)}$, and the unknowns of the element 3-4 are $\phi_3^{(0)}, \phi_4^{(0)}, \phi_3^{(1)}$ and $\phi_4^{(1)}$. To solve the problem, eight equations are necessary. These equations can be divided in three groups: (1) equations based on the CTR, (2) equations based on the connective relations between the elements, and (3) boundary conditions. The equations based on the CTR will rely on the equations (14) and (20) and will generate four equations. The equations based on the connective relations will impose the continuity of the function and its first derivative at the adjacent nodes

$$\phi_2^{(0)} = \phi_3^{(0)}$$

$$\phi_2^{(1)} = \phi_3^{(1)}$$

In addition, two boundary conditions, such as the relations (8) and (9) can be imposed at the ends of the integration domain. Consequently, a total of eight equations will be available from the CTR, connective relations and boundary conditions, such that the constants of the interpolation function CF8 can be determined.

The eigenvalues of straight beams

The governing equation

The deformation analysis of a straight beam, written in the $x-z$ coordinate system, is based on relation (1). Following the procedure used in section 1, $p(x)$ is given by the inertia load (6), where u will be replaced by w:

$$p(x) = w\omega^2 \rho A. \tag{75}$$

Substituting $p(x)$ from equation (75) into (1) yields [5], p. 224

$$w^{(4)} - \beta\, w^{(0)} = 0, \tag{76}$$

where

$$\beta = \omega^2 \frac{\rho A}{EI_z}. \tag{77}$$

Let us replace the displacement w by ϕ, such that the fourth-order ODE (76) becomes

$$\phi^{(4)} - \beta\,\phi^{(0)} = 0 \tag{78}$$

The Complete Transfer Relation

The ODE (78) must be integrated four times. These successive integrations lead to the following four equations that represent the CTR

$$
\begin{cases}
\phi_1^{(3)} - \phi_2^{(3)} - \beta \int_{x1}^{x2} \phi^{(0)}\, dx = 0 \\
-\phi_1^{(2)} + \phi_2^{(2)} + x_1\phi_1^{(3)} - x_2\phi_2^{(3)} - \beta \int_{x1}^{x2} x\,\phi^{(0)}\, dx = 0 \\
2\phi_1^{(1)} - 2\phi_1^{(1)} - 2x_1\phi_1^{(2)} + 2x_2\phi_2^{(2)} + x_1^2\phi_1^{(3)} - x_2^2\phi_2^{(3)} - \beta \int_{x1}^{x2} x^2\,\phi^{(0)}\, dx = 0 \\
-6\phi_1^{(0)} + 6\phi_2^{(0)} + 6x_1\phi_1^{(1)} - 6x_2\phi_2^{(1)} - 3x_1^2\phi_1^{(2)} + 3x_2^2\phi_2^{(2)} + x_1^3\phi_1^{(3)} - x_2^3\phi_2^{(3)} - \beta \int_{x1}^{x2} x^3\,\phi^{(0)}\, dx = 0
\end{cases} \tag{79}
$$

The Concordant Functions

The seventh-degree concordant function CF8

Because equation (78) has been integrated four times, the CTR (79) is based on eight end unknowns $\phi_1^{(3)}, \phi_2^{(3)}$, $\phi_1^{(2)}, \phi_2^{(2)}$, $\phi_1^{(1)}, \phi_2^{(1)}$, $\phi_1^{(0)}$ and $\phi_2^{(0)}$. These eight unknowns are the basic unknowns. In this case the basic concordant function will be a CF8 interpolation function given by equation (55) for which there are no apparent unknowns. Consequently, the relations (55), (62), (63), can be used directly without any elimination procedure.

For a fourth-order ODE, the major modification compared to second-order ODEs is that two additional integrals must be considered in addition to equations (62) and (63). These additional integrals result from the third and fourth equations of (79)

$$
Intx^2\phi 8 = \int_{x1}^{x2} x^2\,\phi^{(0)}\, dx = R_1\phi_1^{(0)} + R_2\phi_2^{(0)} + R_3\phi_1^{(1)} + R_4\phi_2^{(1)} + R_5\phi_1^{(2)} + R_6\phi_2^{(2)} + R_7\phi_1^{(3)} + R_8\phi_2^{(3)} \tag{80}
$$

$$
Intx^3\phi 8 = \int_{x1}^{x2} x^3\,\phi^{(0)}\, dx = S_1\phi_1^{(0)} + S_2\phi_2^{(0)} + S_3\phi_1^{(1)} + S_4\phi_2^{(1)} + S_5\phi_1^{(2)} + S_6\phi_2^{(2)} + S_7\phi_1^{(3)} + S_8\phi_2^{(3)}. \tag{81}
$$

Following a procedure similar to that presented in section 3, a homogeneous system of equations similar to equation (27) results

$$
\Big[\; [A_0] + \beta [A_1] \;\Big] [\bar{\phi}]_8 = [0]. \tag{82}
$$

Note that in this case [A_0] and [A_1] are eight-by-eight matrices and $\left[\bar{\phi}\right]_8$ is an eight-component vector given by relation (60). Four equations of the system of equations (82) are provided by the CTR (79). The other four necessary equations are represented by the boundary conditions. If the beam is simply supported, then both ends displacements and bending moments are zero. By taking into account the equation $w^{(2)} = -M(x)/EI = \phi^{(2)}$ the boundary conditions are

$$
\phi_1^{(0)} = w_1^{(0)} = 0 \;\; ; \;\; \phi_2^{(0)} = w_2^{(0)} = 0 \;\; ; \;\; \phi_1^{(2)} = w_1^{(2)} = 0 \;\; ; \;\; \phi_2^{(2)} = w_2^{(2)} = 0. \tag{83}
$$

The fifteenth-degree concordant function CF16

As shown in section 3, if a concordant function with more terms than those corresponding to the basic unknowns is used, the result is a polynomial eigenvalue problem described by (54). Recall that for the second-order ODE (7), the coefficients of the CF4 interpolation function were calculated using only basic unknowns, without apparent unknowns. In this case, the relation (54) was the same for the CF6 and CF8 interpolation functions. This had a favorable influence on the accuracy of the results. For the fourth-order ODE (78) the coefficients of the CF8 interpolation function can be calculated using only basic unknowns, without apparent unknowns. In this case, equation (54) is the same whether the CF10, CF12, CF14 or CF16 interpolation functions are used. To achieve higher accuracy it is recommended therefore to use the CF16 interpolation function, which is a fifteenth-degree polynomial. Using the same procedure presented in section 3, the integral (62) will now depend on 16 terms

$$\begin{aligned}\beta\, Int\phi 16 = \beta\int_{x1}^{x2} \phi^{(0)}\, dx = \beta\,(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)} + P_5\phi_1^{(2)} + P_6\phi_2^{(2)} + P_7\phi_1^{(3)} + \\ + P_8\phi_2^{(3)} + P_9\phi_1^{(4)} + P_{10}\phi_2^{(4)} + P_{11}\phi_1^{(5)} + P_{12}\phi_2^{(5)} + P_{13}\phi_1^{(6)} + P_{14}\phi_2^{(6)} + P_{15}\phi_1^{(7)} + P_{16}\phi_2^{(7)}).\end{aligned} \tag{84}$$

In this case, eight apparent unknowns must be eliminated. This can be done using the governing equation (78) and its first three derivatives

$$\begin{aligned}\phi^{(4)} - \beta\,\phi^{(0)} &= 0\\ \phi^{(5)} - \beta\,\phi^{(1)} &= 0\\ \phi^{(6)} - \beta\,\phi^{(2)} &= 0\\ \phi^{(7)} - \beta\,\phi^{(3)} &= 0.\end{aligned} \tag{85}$$

By applying them to both ends, the following eight relations result

$$\begin{aligned}\phi_1^{(4)} - \beta\,\phi_1^{(0)} &= 0\\ \phi_2^{(4)} - \beta\,\phi_2^{(0)} &= 0\\ \phi_1^{(5)} - \beta\,\phi_1^{(1)} &= 0\\ \phi_2^{(5)} - \beta\,\phi_2^{(1)} &= 0\\ \phi_1^{(6)} - \beta\,\phi_1^{(2)} &= 0\\ \phi_2^{(6)} - \beta\,\phi_2^{(2)} &= 0\\ \phi_1^{(7)} - \beta\,\phi_1^{(3)} &= 0\\ \phi_2^{(7)} - \beta\,\phi_2^{(3)} &= 0.\end{aligned} \tag{86}$$

Replaced in equation (84), these relations lead to

$$\begin{aligned}\beta\, Int\phi 16 = \beta\int_{x1}^{x2} \phi^{(0)}\, dx = \beta\,(P_1\phi_1^{(0)} + P_2\phi_2^{(0)} + P_3\phi_1^{(1)} + P_4\phi_2^{(1)} + P_5\phi_1^{(2)} + P_6\phi_2^{(2)} + P_7\phi_1^{(3)} + P_8\phi_2^{(3)}) + \\ + \beta^2\,(P_9\phi_1^{(0)} + P_{10}\phi_2^{(0)} + P_{11}\phi_1^{(1)} + P_{12}\phi_2^{(1)} + P_{13}\phi_1^{(2)} + P_{14}\phi_2^{(2)} + P_{15}\phi_1^{(3)} + P_{16}\phi_2^{(3)}).\end{aligned}$$

(87)

A similar procedure applied to Q_i, R_i and S_i, (i=1, 2, … ,16) will lead to equation (54).

A numerical comparison between the DM and the AEM

The DM is based on equation (69), where for the straight beam

$$\left[\overline{\delta}\right] = \left[w_1^{(0)} \quad w_2^{(0)} \quad w_1^{(1)} \quad w_2^{(1)}\right]^T = \left[\phi_1^{(0)} \quad \phi_2^{(0)} \quad \phi_1^{(1)} \quad \phi_2^{(1)}\right]^T. \tag{88}$$

Because only four end unknowns are involved, the DM can only use a third-degree polynomial (Hermite type) to obtain the eigenvalues. This limitation leads to quite poor results, compared to using the CF8 and especially CF16 interpolation functions. The third-degree polynomial will be denoted IF4.

The eigenvalues of a simply supported beam with constant cross-area are given by the exact solution

$$\omega_n = n^2\left(\frac{\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}. \quad (89)$$

If $A=1$, $I=1$, ρ/E=1 and $L=\pi$, the eigenvalues are given by $\omega_1 = 1, \omega_2 = 2^2$, $\omega_3 = 3^2$, $\omega_4 = 4^2$, ... , $\omega_n = n^2$.

The results obtained by using the DM with the IF4 interpolation function are shown in Table 6. The relative errors were calculated according to (39). As known by the users of structural analysis programs, even for the simple case analyzed herein, the beam must be divided in a large number of elements in order to generate accurate results, especially if higher eigenvalues must be computed.

Table 7. Results obtained with IF4 (Hermite interpolation function)

Elements	Relative errors		
NE =	ω_1	ω_2	ω_3
1	**No solution**	**No solution**	**No solution**
2	-7.3 E-3	**No solution**	**No solution**
3	-1.1 E-3	-3.3 E-2	**No solution**
4	-3.1 E-4	-7.3 E-3	-6.3 E-2
5	-1.2 E-4	-2.5 E-3	-1.9 E-2
6	-5.6 E-5	-1.1 E-3	-7.3 E-3
8	-1.9 E-5	-3.0 E-4	-1.8 E-3
10	-5.9 E-6	-1.2 E-4	-6.7 E-4
20	-4.3 E-7	-6.9 E-6	-3.6 E-5
24	-2.0 E-7	-3.3 E-6	-1.7 E-5

The results obtained by using the CF8 and CF16 interpolation functions are shown in Table 8 [1], pp. 113-118. The number of eigenfrequencies was increased up to ω_5, though the analysis was based only on one, two and three elements. The comparison of the results shown in Tables 7 and 8 illustrates that the accuracy of the AEM results is much better than that of the DM results. Let us underline few aspects.

The result obtained for ω_1 using IF4 with twenty elements (20 x 4 = 80 unknowns) is similar to that obtained using CF8 with two elements (2 x 8 = 16 unknowns). If compared to CF16 with two elements, which has also 16 unknowns, the ratio between the relative errors is

$$\frac{error_{IF4}\ (20\,elements, 80\,unknowns)}{error_{CF16}\ (2\,elements, 16\,unknowns)} = \frac{4.3\,E-7}{2.4\,E-9} = 180.$$

If the same number of unknowns is considered, namely 16, then

$$\frac{error_{IF4}\ (4\,elements, 16\,unknowns)}{error_{CF16}\ (2\,elements, 16\,unknowns)} = \frac{3.1\,E-4}{2.4\,E-9} = 1.3*10^5$$

Note also that the error 4.9 E-10 obtained for ω_1 using three elements with CF16 corresponds to ω_1 =1.00000000049. The analytical solution is ω_{1exact} =1, such that the AEM result has ten exact digits.

Table 8. Relative errors for the first five eigenfrequencies obtained with CF8 and CF16

ω	NE = 1 element		NE = 2 elements		NE = 3 elements	
	CF8	CF16	CF8	CF16	CF8	CF16
ω_1	3 E-4	3.4 E-8	6.9 E-7	2.4 E-9	2.8 E-8	4.9 E-10
ω_2	5.2 E-3	-9.7 E-7	3 E-4	3.4 E-8	6.7 E-6	7 E-9
ω_3	5.8 E-1	2.4 E-4	3 E-3	-5.6 E-8	3 E-4	3.4 E-8
ω_4	7.5 E-1	1.3 E-3	4.2 E-3	-9.7 E-7	1.3 E-3	6.3 E-8
ω_5	No solution	8.2 E-2	8 E-2	7.2 E-6	6.1 E-3	-1.8 E-7

Conclusions

The AEM, a new method for the integration of ordinary differential equations, has been developed and implemented to integrate linear and nonlinear ODEs with constant and variable coefficients [1, 3]. The AEM has been applied herein to compute accurately and efficiently the axial vibration of trusses and the transverse vibration of straight beams. The AEM can model two- and three-dimensional systems of beams and trusses with constant and variable cross section, as well as beam deformation due to shear forces [1]. The results presented herein were limited to two-dimensional trusses and straight beams.

The truss element solved a second-order ODE, for which four basic unknowns must be used. The basic concordant function CF4 was therefore a four-term, third-degree polynomial. The straight-beam element solved a fourth-order ODE, for which eight basic unknowns must be used. The basic concordant (or interpolation) function CF8 is an eight-term, seventh-degree polynomial.

As shown herein, the concordant functions have an essential role in the accurate element method, because they allow the use of high-degree polynomials without increasing the number of the end unknowns. An important step in obtaining the concordant functions is the calculation of the inverse matrix $[A]^{-1}$. This step may become time consuming, especially for the very high-degree concordant functions. The methodology presented here is a simplified version intended to facilitate understanding the AEM. The methodology implemented in the code, however, uses concordant functions written for a natural (non-dimensional) coordinate system. In this last case the inverse matrix $[A]^{-1}$ is always the same, being calculated only once. The inverse matrices $[A]^{-1}$ are given several concordant functions in [1].

This paper was focused on the solution of beam and trusses free vibration. The AEM allowed us to obtain a large number of high frequencies using a small number of elements, thereby reducing the computational time. The results obtained using the AEM were compared against finite element results obtained using ANSYS. For both trusses and beams, the accuracy of the eigenvalues computed using the AEM was several orders of magnitude higher than that of the finite element analysis, while the computational time was approximately the same.

Because the strategies of AEM and DM are similar, the AEM elements can be included without special problems in a structural analysis program, such as ANSYS. The use of high-degree concordant functions can be handled by a subprogram that will automatically eliminate the apparent unknowns, returning to the main program the Complete Transfer Relation including only the small number of the basic unknowns.

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