Theoretical Analysis for Constitutive Parameters of the Periodic Electric Resonator Metamaterials

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Abstract—We begin with Maxwell’s equations and put the thin electric resonance dielectric sheet equivalent into electric surface current. Using periodic boundary conditions and the principle of superposition, we can get the dispersion relation and Bloch impedance of periodic electric resonator metamaterials. Base on the dispersion relation and Bloch impedance, we can obtain the theoretical analytical expressions of Bloch constitutive parameters. Since we consider the impact of the magnetic anti-resonance, we find theoretical predictions match the retrieval results very well, which showed that the analytical expression derived in the paper is extremely effective.

Keywords—Periodic structure; Electric resonance; Bloch constitutive parameters; Surface electric current

I. INTRODUCTION

Artificially structured electromagnetic metamaterials have received considerable attention in the recent decades, for their ability to exhibit electromagnetic properties not found in conventional materials. These metamaterials have simultaneously negative permittivity and permeability over a certain frequency band and can be constructed artificially to show several exotic properties that cannot be easily achieved using natural materials [1-9]. Since the demonstration of an artificial medium with negative refractive index in 2001[3], various types of structures have been designed and verified to be metamaterials [10,11], besides negative refraction property, many other properties of metamaterials have now been extensively investigated [12,13].

In order to interpret the artificial electromagnetic of various metallic structures, researchers presented various models which are all based on the RLC-resonance of the metallic structure [14-17]. The RLC circuit model presented a clear and instructive physics paradigm for the metallic structured metamaterials, and became a powerful tool for the prediction and design of metamaterials. A single metallic metamaterials inclusion can be considered as an LC resonant circuit with its inductance and capacitance. These resonators can collectively exhibit macroscopically observed effective values of permittivity or permeability. Various forms of resonant inclusions have been introduced to date, such as the split-ring resonators (SRR) for a magnetic response [14], and the electric-LC (ELC) resonator which has strong electric resonance [18]. When the resonance of the metal structure occurs, there is an antiresonance [14,19]. Under normal circumstances, the antiresonance is very weak, and has little effect on the electromagnetic parameters.

To examine the constitutive parameters (permittivity and permeability) of metamaterials, various methods have been proposed for retrieval of these properties when they are exposed to an electromagnetic stimulus [20]. Scattering (S-) parameter material extraction methods has been analyzed by numerous researchers [21-27], the methods allow analyses of both numerical simulation and experiment. These methods can be used to form an initial metamaterial design and develop a working intuition, but do not predict the ultimate frequency-dependent form that the actual parameters usually take. The complicated, frequency-dependent forms of the constitutive parameters can be described by a set of relatively simple analytical expressions. These expressions provide useful insight and can serve as the basis for more intelligent interpolation or optimization schemes [19]. Literature [28] using a transfer-matrix formalism applied to a one-dimensional periodic array of thin, resonant, dielectric sheets, and deduce the analytical expression of constitutive parameters with effective medium theory. But the analytical method have some deficiencies, the anti-resonance is not being taken into account in the curve fitting. As it can be seen in the figure of ELC constitutive parameters from the literature, there are some discrepancy between the simulations and analytical formulas, especially toward higher frequencies.

To decrease the discrepancy of the analytical expression, we take into the impact of the magnetic anti-resonance—average relative permeability \( \bar{\mu} \). We begin with Maxwell’s equations and put the thin electric resonance dielectric sheet equivalent into electric surface current under some conditions. Using periodic boundary conditions and the principle of superposition, we can get the dispersion relation and Bloch impedance of periodic electric resonator metamaterials. By the dispersion relation and Bloch impedance, we can obtain the theoretical analytical expressions of Bloch constitutive parameters. Since we take into the impact of \( \bar{\mu} \), we find
theoretical predictions match retrieval results very well, which showed that the analytical expression derived in the paper is extremely effective.

II. THEORY

The model followed in this paper will be to consider a wave propagating in an infinite, periodic array of thin, polarizable sheets. The geometry of the model is shown in Fig.1. A series of polarizable, planar sheets, of width $h$, is spaced apart with periodicity $d$. We assume that an electromagnetic wave propagates in the direction along the z-axis, and the electric surface current density $J_s$ along the x-axis.

![Fig.1 Periodic electric resonance artificial medium sheets. The sheets have thickness $h$ and are spaced a distance $d$ apart. An electromagnetic wave is assumed to propagate in the direction along the z-axis, and the electric surface current density $J_s$ along the x-axis.

According to the Maxwell magnetic field curl equation $\nabla \times \mathbf{H} = j\omega \varepsilon_0 \mathbf{J}$, we assume the electric volume current density

$$\mathbf{J} = j\omega \varepsilon_0 \mathbf{S} \mathbf{E}$$

in which $\varepsilon_0$ is the relatively permittivity of the electric resonance artificial medium sheets, $\varepsilon_0$ is the permittivity of the air, $\omega$ is the angular frequency. Since we are considering the sheets very thin, we take the limit $h \rightarrow 0$, we consider the electric volume current density flows on a plane. The sheets are spaced apart with periodicity $d$, in the coordinates of the z-axis can be expressed as $md$ ($m$ is an integer, the range is $(-\infty, \infty)$). Applying Eq.(1), we arrive the electric surface current density at $z = md$

$$J_s(z) = j\omega \varepsilon_0 h \mathbf{E}(z) \quad z = md \quad (2)$$

If the electrical resonant medium sheet have a resonance, $\varepsilon_r \rightarrow \infty$, which leads $\omega \varepsilon_r \varepsilon_0 \rightarrow \infty$ and $\omega \varepsilon_r \varepsilon_0 h \neq 0$. Thus $J_s(z) \neq 0$ at $z = md$, we can use Maxwell magnetic field curl equation $\nabla \times \mathbf{H} = j\omega \varepsilon_0 \mathbf{J}$, and the electric surface current density $J_s$ is equivalent to the electric volume current density $\mathbf{J}$.

The electromagnetic wave propagates in the direction along the z-axis and the electric surface current density $J_s$ in the direction along the x-axis. Using periodic boundary conditions, at $z = md$ we get the expression of electric surface current density

$$J_s(z) = J_{s0} e^{-j\beta z} \hat{x} \quad z = md \quad (3)$$

in which $J_{s0}$ is the amplitude of the electric surface current density at $z = 0$. $\beta$ is the propagation constant of periodic electrical resonance material.

We assume that there is only one electric surface current density at $z = md$, the electric surface current density will produce an electric field in the space can be expressed as

$$\mathbf{E}^- = A e^{j k_0 (z - md)} \hat{x} \quad z < md \quad (4)$$

$$\mathbf{E}^+ = A e^{j k_0 (z - md)} \hat{x} \quad z > md \quad (5)$$

Which $A$ is an arbitrary amplitude constant, $k_0 = \omega \sqrt{\varepsilon_0 \mu_0}$ is the propagation constant of air ($\mu_0$ is the permeability of air). According to Maxwell electric field curl equation $\nabla \times \mathbf{E} = -j\omega \varepsilon_0 \mathbf{H}$, using Eq.(4) and Eq.(5), we arrive

$$\mathbf{H}^- = \frac{A}{\eta_0} e^{-j k_0 (z - md)} \hat{y} \quad z < md \quad (6)$$

$$\mathbf{H}^+ = \frac{A}{\eta_0} e^{j k_0 (z - md)} \hat{y} \quad z > md \quad (7)$$

where $\eta_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the wave impedance of air, $z = md$ substituting into Eq.(3), Eq.(6) and Eq.(7), using magnetic field boundary conditions $\hat{n} \times (\mathbf{H}^- - \mathbf{H}^+ ) = J_s (\hat{n}$ and $\hat{z}$ have the same direction, $\hat{n} = \hat{z}$) at $z = md$, we obtain

$$A = -\frac{\eta_0}{2} J_{s0} e^{-j \beta md} \quad (8)$$

Applying Eq.(8), we can write Eq.(4)-Eq.(7) as

$$\mathbf{E}^- = -\frac{\eta_0}{2} J_{s0} e^{-j \beta md} e^{j k_0 (z - md)} \hat{x} \quad z < md \quad (9)$$

$$\mathbf{E}^+ = -\frac{\eta_0}{2} J_{s0} e^{j \beta md} e^{-j k_0 (z - md)} \hat{x} \quad z > md \quad (10)$$

$$\mathbf{H}^- = \frac{1}{2} J_{s0} e^{-j \beta md} e^{-j k_0 (z - md)} \hat{y} \quad z < md \quad (11)$$

$$\mathbf{H}^+ = \frac{1}{2} J_{s0} e^{j \beta md} e^{j k_0 (z - md)} \hat{y} \quad z > md \quad (12)$$
These expressions are the electromagnetic waves which generated by the electric surface current at surface $z = nd$.

In order to solve the Bloch impedance expressions and the dispersion relations of the electrical resonance artificial medium, we should solve the total electromagnetic field generated by each sheet. According to the superposition principle, the total electromagnetic field at surface $z = (n + 1/2)d$ is linear superposition of the electromagnetic waves generated by each sheet. Electric surface current is the equivalent of electrical resonance artificial medium sheet, yet when electrical resonance occurs, there is a magnetic anti-resonance. With the occurrence of the magnetic anti-resonance, the permeability of the electrical resonance artificial medium sheet will change.

According to the effective medium theory, in the air space periodic is much smaller than the wavelength, the periodic electric resonator artificial medium sheet can be equivalent into a homogeneous medium. Consider the influence of the magnetic anti-resonance, we should introduce a parameter of homogeneous medium average relative permeability $\bar{\mu}$. (It is the average relative permeability of air and electrical resonance sheet). Therefore, with $k_0 \sqrt{\bar{\mu}}$ and $\eta_0 \sqrt{\bar{\mu}}$, substitute the propagation constant $k_0$ and the wave impedance $\eta_0$ in Eq.(9-12). We can get total magnetic field

$$H_{total,n+1/2} = \sum_{m=-\infty}^{n} H^+ + \sum_{m=n+1}^{\infty} H^-$$

and total electric field

$$E_{total,n+1/2} = \sum_{m=-\infty}^{n} E^+ + \sum_{m=n+1}^{\infty} E^-$$

Using Eq.(13) and Eq.(14), we arrive at the simple expression of propagation constant

$$\beta = \frac{2}{d} \arcsin \left( \frac{k_0d}{2} \sqrt{\bar{\mu}_r \left( 1 + \frac{h}{d} \bar{\varepsilon}_r \right)} \right)$$

Using Eq.(20) and Eq.(21), we can write Eq.(15) as

$$Z_B = \frac{E_{total,n} \cdot \hat{x}}{H_{total,n+1/2} \cdot \hat{y}}$$

Similarly, we can get total electric field $E_{total,n}$ at $z = nd$.
\[ Z_B = \frac{\mu_0 \mu_B}{\varepsilon_0 \varepsilon_B} \]  
\[ \beta = \omega \sqrt{\mu_0 \mu_B \varepsilon_0 \varepsilon_B} \]  

From Eq.(22)-Eq.(25), we find directly the approximate expressions for the Bloch constitutive parameters

\[ \mu_a = \frac{k_d}{2 \sqrt{\mu_r}} \sqrt{1 + \frac{h}{d} \varepsilon_r} \left( \frac{k_d}{2} \sqrt{\mu_r \left( 1 + \frac{h}{d} \varepsilon_r \right)} \right) \]  
\[ \varepsilon_B = \frac{k_i d}{2} \sqrt{1 - \mu_r \left( 1 + \frac{h}{d} \varepsilon_r \right) \left( \frac{k_i d}{2} \right)^2} \]  

From Eq.(26) and Eq.(27), we can observe the important of the magnetic anti-resonance \( \mu_r \). If we ignore the magnetic anti-resonance, there will be some discrepancy between retrieval results and analytical theory. Using the parameters \( (\varepsilon_r, \mu_r, h, d, \omega) \) of the periodic electric resonator artificial medium, we can calculate the Bloch constitutive parameters. Thus, Eq.(26) and Eq.(27) provide a useful description for the Bloch constitutive parameters of the periodic electric resonator artificial medium, the analytic theory will be compared with numerical retrievals in next section.

### III. COMPARISONS AND DISCUSSIONS

In order to validate the analytic theory of the periodic electric resonator artificial medium, we consider the electric-LC (ELC) resonator structures for analysis, comparing their retrieved effective index and Bloch parameters to those predicted analytically using the equations derived in the preceding sections. The \( S \)-parameter retrieval method we present here makes use of periodic boundary conditions on the surfaces of a single unit cell along the propagation direction, such that a structure effectively infinite in the lateral directions is simulated. The numerical simulations are performed using HFSS (Ansys), a commercial electromagnetic mode solver that is based on the finite element method [28].

For the example in this section, we follow the procedure of computing the scattering (\( S \)) parameters for the structures utilizing the driven solution in HFSS, with input and output ports defined on two opposing faces of the unit cell. A combination of periodic and electric-magnetic boundary conditions is used both to define the polarization of the wave and to simulate an infinite lattice perpendicular to the direction of propagation [21, 23]. From the computed \( S \) parameters, the Bloch constitutive parameters can be found by applying a standard retrieval process [21].

The average and effective constitutive parameters have often been used as an approximate description of metamaterials in previous research, where Drude-Lorentz and similar models have been applied [28-40]. Yet it is well known that there are significant deviations from these ideal forms when numerical retrievals are performed on simulated or measured data. In fact, the Drude-Lorentz models are accurate, causal descriptions of metamaterials in the limit that spatial dispersion is not a factor (i.e., electro- or magnetostatic limits). In the presence of periodicity, though, the ideal forms are modified in the manner described above [28]. The ELC structure shown in Fig.2 possesses an average permittivity is of the form

\[ \varepsilon_r = \varepsilon_a \left( 1 - \frac{Ff^2}{f^2 - f_{oe}^2 + j\gamma f} \right) \]  

![Fig. 2](image_url)  
(a) ELC structure. The wave is assumed to propagate between the two patterned surfaces, polarized such that the electric field excites the ELC. (b) ELC consists of a conducting ring and the substrate. The substrate is Flame Resistant 4 (FR4), a type of material used for making a printed circuit board, whose relative permittivity is \( \varepsilon = 4.4 + 0.001i \) and thickness 0.2mm. The dimensions are as=3mm, b=2.6mm, c=d=g=0.2mm, f=0.02mm and e=0.7mm. The printed circuit board thickness is assumed as 17\( \mu \)m. Periodic boundary conditions are applied on the sides of the unit cell (perpendicular to the propagation direction) to simulate a metamaterial array infinite in the lateral dimensions.
In which $f_0$ is the electric resonant frequency, and $\gamma$ is the loss factor [19]. Based on the ideal forms, we can now calculate the effective permittivity using Eq.(16) and Eq.(17).

Fig.3 compares the predicted parameters for the ELC unit cell with those from direct simulation. As can be seen, there is remarkably good overall agreements between the analytic theory and the simulations. The real and imaginary parts of the retrieved parameters are shown as the solid black curves, respectively. The dashed red and green curves correspond to

![Graph](image-url)

**Fig.3.** Comparison of theoretical prediction results and retrieval results from the scattering parameters $S$ for the ELC structure. (a) and (b) are the Bloch constitutive parameters, (c) and (d) is the propagation constant and the relative wave impedance. The parameters using in the theoretical calculation are $f_0=9.4\text{GHz}$, $\varepsilon_r=0.96$, $\mu_r=0.945$, $\gamma=0.16\text{GHz}$, $h=0.0105$ mm, and $F=0.1893$. Matter whether in the retrieval or the theoretical result [19], this phenomenon is called the electrical resonance and magnetic anti-resonance, and 11.4 GHz is the electric resonance frequency. This phenomenon can be clearly explained by the general theory of effective media given in Eq.(26) and Eq.(27). Our analysis take into account of the magnetic antiresonance and introduce the average relative permeability $\bar{\mu}$, therefore, Eq.(26) and Eq.(27) are well fitted the trend of the ELC resonator constitutive parameters, as can be seen from Fig.3. Literature [28] ignores the magnetic antiresonance, which leads to a large discrepancy between the simulations and analytical formulas of ELC resonator constitutive parameters.

**IV. CONCLUSION**

This paper provides a descriptive approach and a explanation for the constitutive parameters of the periodic electric resonator metamaterials. According to Maxwell’s equations, the thin electric resonance dielectric sheet equivalent into electric surface current, and then using the superposition principle to obtain the total electromagnetic fields which generated by all electric surface currents, whereby the analytical expressions of the Bloch permittivity and Bloch permeability can be deduced. In the derivation process, taking into the impact of magnetic antiresonance, so the theoretical results match the retrieval results very well. We have compared and analyzed the behavior of ELC structures by theoretical predictions, using the general theory and numerical retrieval simulations. The analysis provides profound understanding of the electric resonance, and the magnetic antiresonance observation. Although the analytical expression is complicated, it will give guidance for synthesis and design of metamaterial structures.
REFERENCES


