Finite Element Determination of Critical Zones in Composite Structures

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Abstract
Modeling and structural analysis of composite materials (CM) with complicated microstructure is very labor-consuming job. In the current paper a new method of composite structures analysis is suggested. It enables to determine with high degree of accuracy critical zones of potential failure where critical stressed state is achieved and only then analyze in details micro stress fields.

Main steps of the method are:
1. Homogenization of composite structure - computation of effective thermo-mechanical characteristics and failure surface tensors.
2. Macro analysis of the construction stressed state and determination of critical zones.
3. Sequential heterogenization in critical zones and determination of micro stresses.

Suggested approach is based on a new form of Tsai-Wu tensorial-polynomial failure criterion construction and application of up-to-date computational methods implemented into ANSYS. This approach differs from the one considered in the paper of Tsai and Wu [2] by numerical but not experimental determination of failure surface tensors components for the periodical cell.

After that, during macro analysis of the homogenized structure tensorial-polynomial criterion is applied to determine critical zones. The criterion is satisfied when the point that characterizes stressed state of the whole periodical cell in the space of averaged stresses is located outside effective failure surface.

To construct effective failure surface the macro in ANSYS Parametric Design Language (APDL) was developed that enables to calculate about 300 points of the failure surface. Every point is a result of solution of problem for the periodical cell with boundary conditions corresponding to the loading type. The number of points calculated in the present work allows creating 4-th order surfaces. It corresponds to keeping items containing 8-th order failure surface tensors in the Tsai-Wu criterion.

Verification of the method developed is done by solving plane stress problems and determination of zones where critical stressed state is achieved:
- compression of fibrous periodical composite plate;
- complex loading of fibrous periodical composite plate with central hole.

Etalon solution is also obtained by direct finite element modeling of the construction containing all microstructure in order to estimate "quality" of the results obtained with use of new method. It is determined that method developed enables to define efficiently all critical zones in the construction and with high degree of accuracy compute micro stresses.

Introduction
Modeling and structural analysis of composite materials with consideration of complex microstructure is very labor-consuming job that includes many problems. One of them is determination of zones where critical stress state is observed. Even despite recent achievements in computer techniques and science-intensive analysis software the solution of this problem involves difficulties. That’s why the necessity is arising of the new methods development that could solve such problems with minimum time and labor.

In the current paper the new method of stressed state of composite materials is suggested. This method enables one to determine with high degree of accuracy critical zones of potential failure and then analyze in details micro stresses fields.

Main steps of the method are:
1. Homogenization of composite structure - computation of effective thermo-mechanical characteristics and failure surface tensors.

2. Macro analysis of the construction stressed state and determination of critical zones.

3. Sequential heterogenization in critical zones and determination of micro stresses.

Suggested approach is based on a new form of Tsai-Wu tensorial-polynomial failure criterion construction and application of up-to-date computational methods implemented into ANSYS.

Verification of the developed method of potential failure zones where critical stressed state is reached is performed for a number of plane stress problems:

1. Plate made of periodical fiber composite under compression;

2. Plate made of periodical fiber composite with central hole under complex loading.

In order to verify the obtained results by direct FE modeling the reference solutions were obtained with full account of real microstructure of the composite material. It was found that the developed method enables to find all critical zones of potential failure in the construction and with high degree of accuracy calculate micro stresses.

To illustrate the suggested method unidirectional fiber composite was considered. Material of the matrix is brittle polymer, fiber – steel. Periodical cell is presented in Figure 1, where $a = 25 \mu m$, $b = 20 \mu m$, $c = 15 \mu m$. Volume concentration of the fiber is $V_f = 0.3534$. Elastic characteristics of the materials are listed in Table 1.

![Figure 1. Periodical cell](image)

<table>
<thead>
<tr>
<th></th>
<th>$E$ (GPa)</th>
<th>$v$</th>
<th>$\sigma^0$ (MPa)</th>
<th>$\sigma^p$ (MPa)</th>
<th>$\sigma^c$ (MPa)</th>
<th>$\sigma^t$ (MPa)</th>
<th>$Y$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>matrix</td>
<td>3.4</td>
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<td>–</td>
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<td>–</td>
<td>–</td>
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</tr>
</tbody>
</table>

**Statement of the problem**

During the development of criterion by Tsai-Wu [2] computer techniques didn’t allow to carry out complex numerical simulations and composite failure could be analyzed only during experiments. At present time computers and software give possibility to solve complex problems with account of physical and geometrical non-linearity. Nevertheless, determination of critical zones in composite materials with complex microstructure is still very consuming task. That’s why it is suggested in the present work to turn to homogeneous material and analyze failure process for it.

To analyze failure it is suggested to use tensorial-polynomial criterion that has the following form (Tsai-Wu, [2]):

$$ ^2 F \cdot \sigma + \sigma^4 \cdot ^4 F \cdot \sigma + [\sigma^6 \cdot ^6 F \cdot \sigma] \cdot \sigma + [\sigma^8 \cdot ^8 F \cdot \sigma] \cdot \sigma + ... = 1 $$

(1)

where $^i F$ – failure surface tensor of i rank.

In the coordinate form the criterion is looking as the following:

$$ ^{ijkl} F_{ij} \sigma_{ij} + ^{ijkl} F_{ijkl} \sigma_{ijkl} + ^{ijkl} F_{ijkl} \sigma_{ijkl} + ^{ijkl} \sigma_{ijkl} + ^{ijkl} \sigma_{ijkl} + \sigma_{ijkl} + \sigma_{ijkl} + \sigma_{ijkl} + ... = 1 $$

(2)
where $F_{ijklmnsp}$ – components of 8-th rank failure surface tensor ($i,j,k,l,n,m,p,s = 1,2$).

Geometrically the failure criterion can be interpreted as some surface in the space of stresses, i.e. condition of failure is satisfied when defined stress vector crosses this failure surface in point A (Figure 2).

Figure 2. Geometric interpretation of failure condition

The suggested way of failure surface construction differs from the considered in Tsai-Wu’s work by the method of failure surface tensor components determination. In Tsai-Wu’s paper experimental approach was utilized, while in the present work failure surface tensor components are calculated based on numerical analysis of periodical cell that is carried out with use of FE software ANSYS and APDL.

In general failure surface is being built in 6D and it is closed. The absence of closure means infinite ultimate stress in some direction. In the current work it is suggested to consider 2D stressed state so that failure surface would be built is 3D space $<\sigma_{11}><\sigma_{22}><\sigma_{12}>$.

Algorithm of composite structure analysis has the following steps:
1. Switch from heterogeneous model of composite material to homogeneous (determination of $C^*$ – effective elastic moduli tensor);
2. Construction of closed failure surface (in stresses space) according to tensorial-polynomial criterion for homogeneous material based on heterogeneous periodical cell (estimation of $\mathbf{F}(<\sigma>)$, $\mathbf{F}(<\sigma>)$... – effective failure surface tensors);
3. Usage of the constructed failure surface for the detailed analysis of heterogenization zones (micro stress concentration zones, ...);
4. Sequential heterogenization in the zones of criterion satisfaction to obtain more exact solution.

**Switch from heterogeneous to homogeneous model of the composite material**

Switch from heterogeneous model of the composite material to homogeneous one was carried out by direct homogenization method. This method includes:
1. Computation of effective Young’s modulus $E_1^*$ based on volume concentration of fibers $V_f$:

   \[ E_1^* = E_1 V_f + E_m(1-V_f) \]

2. Two problems of transverse tension of the periodical cell (plane strain) to calculate effective Young’s moduli $E_1^*$ and $E_2^*$, and also Poisson’s coefficients.
3. The problem of periodical cell transverse shear (plane strain) to calculate effective shear modulus $G_{12}^*$.
4. Two problems of periodical cell longitudinal shear (anti-plane strain) to calculate effective shear moduli $G_{23}^*$ and $G_{31}^*$.

2.1. Boundary conditions for the first problem of transverse tension:

   \[
   x_1 = 0.5 \quad a: u_1 = u_2 = 0; \quad x_1 = 0 \quad : u_1 = 0, \sigma_{12} = 0 \\
   x_2 = 0.5 \quad b: u_3 = 0, \sigma_{12} = 0; \quad x_2 = 0 \quad : u_3 = 0, \sigma_{13} = 0
   \]
2.2. Boundary conditions for the second problem of transverse tension:

\begin{align*}
    x_i = 0.5 \ a : u_i = 0, \sigma_{ii} = 0; \ x_j = 0 : u_j = 0, \sigma_{jj} = 0 \\
    x_i = 0.5 \ b : u_i = u'_i, \sigma_{ii} = 0; \ x_j = 0 : u_j = 0, \sigma_{jj} = 0
\end{align*}

Using effective determinate relations for macro-orthotropic heterogeneous media and connections between effective Young’s moduli and Poisson’s coefficient the following system of equations is formed:

\begin{align*}
    &E_1^* \varepsilon_{11}^0 = \left( \sigma_{11}^{(0)} \right) - \nu_{12}^* \left( \sigma_{22}^{(0)} \right) - \nu_{13}^* \left( \sigma_{33}^{(0)} \right) \\
    &0 = \left( \sigma_{22}^{(0)} \right) - \nu_{21}^* \left( \sigma_{11}^{(0)} \right) - \nu_{23}^* \left( \sigma_{33}^{(0)} \right) \\
    &0 = \left( \sigma_{33}^{(0)} \right) - \nu_{31}^* \left( \sigma_{11}^{(0)} \right) - \nu_{32}^* \left( \sigma_{22}^{(0)} \right) \\
    &0 = \left( \sigma_{12}^{(2)} \right) - \nu_{12}^* \left( \sigma_{22}^{(2)} \right) - \nu_{13}^* \left( \sigma_{33}^{(2)} \right) \\
    &E_2^* \varepsilon_{22}^0 = \left( \sigma_{22}^{(2)} \right) - \nu_{21}^* \left( \sigma_{12}^{(2)} \right) - \nu_{23}^* \left( \sigma_{33}^{(2)} \right) \\
    &0 = \left( \sigma_{33}^{(2)} \right) - \nu_{31}^* \left( \sigma_{12}^{(2)} \right) - \nu_{32}^* \left( \sigma_{22}^{(2)} \right)
\end{align*}

The solution of the current system of equations:

\begin{align*}
    &E_i^* = \frac{E_i a_{ii}}{E_i^* \left( \sigma_{ii}^{(i)} \right) a_{ii} + a_{ii}^2} \\
    &v_{21}^* = \frac{E_i^* \left( \sigma_{12}^{(i)} \right) a_{12} - E_i^* \left( \sigma_{21}^{(i)} \right) a_{21}}{E_i^* \left( \sigma_{21}^{(i)} \right) a_{21}}, \\
    &v_{23}^* = \frac{E_i^* a_{23}}{E_i^* a_{21}}, \\
    &v_{32}^* = \frac{a_{32}}{a_{21}},
\end{align*}

where \( a_i = \left( \sigma_{ii}^{(i)} \right) - \left( \sigma_{ii}^{(i)} \right) a_i \), \( a_i = -a_i \).

3. Boundary conditions for the problem of periodical cell transverse shear (computation of \( G_{12}^* \)):

\begin{align*}
    x_i = 0.5 \ a : u_i = 0, \sigma_{ii} = 0; \ x_j = 0 : u_j = 0, \sigma_{jj} = 0 \\
    x_i = 0.5 \ b : u_i = u'_i, \sigma_{ii} = 0; \ x_j = 0 : u_j = 0, \sigma_{jj} = 0
\end{align*}

Then:

\begin{align*}
    &G_{12}^* = \frac{\left( \sigma_{12} \right)}{\left( \gamma_{12} \right)}, \\
    &\left( \gamma_{12} \right) = \frac{u_i}{b} (\frac{b}{2})
\end{align*}

4.1. Boundary conditions for the first problem of the periodical cell longitudinal shear (computation of \( G_{23}^* \)):

\begin{align*}
    x_i = 0.5 \ a : \sigma_{ij} = 0; \ x_j = 0 : \sigma_{ij} = 0 \\
    x_i = 0.5 \ b : u_i = u'_i, \sigma_{ii} = 0; \ x_j = 0 : u_j = 0
\end{align*}

Then:

\begin{align*}
    &G_{23}^* = \frac{\left( \sigma_{23} \right)}{\left( \gamma_{23} \right)}, \\
    &\left( \gamma_{23} \right) = \frac{u_i}{b} (\frac{b}{2})
\end{align*}

4.2. Boundary conditions for the second problem of the periodical cell longitudinal shear (computation of \( G_{31}^* \)):
\[
x_1 = 0.5 \ a : u_1 = u_1^0; \ x_2 = 0 : u_2 = 0
\]
\[
x_1 = 0.5 \ b : \sigma_{11}^0 = 0; \ x_2 = 0 : \sigma_{22}^0 = 0
\]

Then:
\[
G_{11} = \frac{\left\langle \sigma_{11} \right\rangle}{\left\langle \gamma_{11} \right\rangle}, \quad \left\langle \gamma_{11} \right\rangle = \frac{u_0}{a \left(\frac{\pi}{2}\right)}
\]

In Table 2 effective characteristics obtained by this algorithm are listed.

Table 2

<table>
<thead>
<tr>
<th></th>
<th>(E_1) (GPa)</th>
<th>(E_2) (GPa)</th>
<th>(E_3) (GPa)</th>
<th>(\nu_{12})</th>
<th>(\nu_{13})</th>
<th>(\nu_{23})</th>
<th>(G_{12}) (GPa)</th>
<th>(G_{13}) (GPa)</th>
<th>(G_{23}) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>7.75</td>
<td>8.64</td>
<td>76.4</td>
<td>0.410</td>
<td>0.034</td>
<td>0.037</td>
<td>2.07</td>
<td>2.40</td>
<td>2.83</td>
</tr>
</tbody>
</table>

**Construction of the failure surface**

Construction of the closed failure surface for homogeneous material contains the following steps:

1. Creation of representative points.
2. Approximation of the created representative points with closed surface in the stress space with use if least squares method.

**Creation of representative points**

To calculate representative points coordinates it is necessary to solve a number of problems for the periodical cell with various boundary conditions. Averaged micro stressed state will be calculated and representative points coordinated will be calculated based on it.

In every problem periodical material with varied Hashin – Rosen boundary conditions [8] is considered. General statement of these problems is presented in Figure 3, where \(l\) – characteristic value of the boundary displacement; \(u, v, w\) – integer numbers.

![Figure 3. Hashin-Rosen boundary conditions](image)

In the current work only plane stressed state is considered. Infinite strains tensor has the following structure:

\[
\tilde{\varepsilon} = \begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} \\
\varepsilon_{yx} & \varepsilon_{yy}
\end{pmatrix} = \begin{pmatrix}
\frac{u}{c} & \frac{1}{c} \\
\frac{w}{c} & \frac{v}{c}
\end{pmatrix}
\]

where \(c\) – fiber diameter,
$l$ – characteristic value of the boundary displacement,

$u, v, w$ – integer numbers.

Every problem is characterized by its own combination of boundary conditions and, consequently, its own combination of values of $u, v, w$. Such boundary conditions enable to realize all potential stressed states of periodical cell. Characteristic value of boundary displacement $l$ is chosen small enough not to reach ultimate stress in any periodical cell.

Representative point is created by averaging of micro stress field after the solution of problem with corresponding boundary conditions. Averaging of the stress fields gives representative point $A (<\sigma_{11}^1, <\sigma_{22}^1>):$

$$
<\sigma_{ij}^{(A)} >= \frac{1}{S_0} \int_{S_0} \sigma_{ij} dS 
$$

where $<\sigma_{11}^1>, <\sigma_{22}^1>$ - averaged field of stress tensor components $\sigma_{11}, \sigma_{22}$,

$S_0$ - central cell surface.

To reach critical stress it is required to find out correction factor (multiplier) $K$ that turns stresses into critical. Switch from point $A$ into point $B$ is realized by means of correction factor:

$$
<\sigma_{ij}^{(B)} >= K <\sigma_{ij}^{(A)}>
$$

where $i,j = 1, 2$.

Scheme of representative point creation is presented in Figure 4.

**Figure 4. Scheme of representative point creation**

Multiplied by factor $K$ maximum principal stresses in the material become equal to critical:

$$
K = \inf \left( \sigma_1^{\text{max}} / \sigma_1, \sigma_2^{\text{max}} / \sigma_2, \sigma_1 / \sigma_2, \sigma_2 / \sigma_1, \sqrt{\sigma_1^2 + \sigma_2^2} / \sigma_{\text{int}} \right),
$$

where infimum is calculated over all nodes of the matrix,

$\sigma_1, \sigma_2$ - principal stresses,

$$
\sigma_1^{\text{max}} = \frac{1}{2} \max \{|\sigma_1 - \sigma_2|\}, \quad \sigma_{\text{int}} = \frac{1}{\sqrt{2}} |\sigma_1 - \sigma_2|.
$$

Consider creation of one of representative points (for plane stress) with parameters $w = 2$, $u = k = 1$. FE mesh is presented in Figure 5. In this case infinite strains tensor has the following structure:
\[
\mathbf{\varepsilon} = \begin{pmatrix}
0.000066 & 0.000066 \\
0.000066 & 0.000133
\end{pmatrix}
\]

\(NDF = 31358\)

**Figure 5. FE mesh for creation of one representative point**

In Figure 6 \(\sigma_{11}\) stress tensor component field is presented for the whole model and central cell. Kinematic boundary conditions of Hashin – Rosen doesn’t satisfy the condition of static compatibility of the deformed periodical cells. To satisfy these conditions in the central periodical cell it is suggested to add layers of periodical cells surrounding the central cell. At this the necessity of the following researches is arising:

1. Practical convergence depending on the number of degrees of freedom (NDF).
2. Practical convergence depending on the number of periodical cells layers surrounding the central cell.

Obtained microstress \(\sigma_{11}\) field for the whole model.  

Microstress \(\sigma_{11}\) in the selected unit cell:

- cell for averaging  
- zone of periodic solution

Boundary cells – zone of double periodicity loss

**Figure 6. Stress field \(\sigma_{11}\) of one representative point**

1. The location of point F, for which the convergence was investigated, is shown in Figure 7, and the convergence plot for matrix stress \(\sigma_{11}^{(P)}\) depending on the number of degrees of freedom is shown in Figure 8. For Figure 7 point F is situated in the middle between points G and H. For Figure 8 FE mesh corresponding to the point marked with red circle is taken for the following analysis. Averaging of stress fields in the central cell gives the following results (point A):

\(<\sigma_{11}> = 0.117 \text{ MPa} \quad <\sigma_{22}> = 0.232 \text{ MPa} \quad <\sigma_{12}> = 0.014 \text{ MPa}\)
Field of principal stresses $\sigma_1$ in the central cell is presented in Figure 9. For the current statement of the problem correcting factor $K$ is calculated in the marked points, because there $\sigma_1$ reaches its maximum. So, the failure occurs at tension.

Thus, the correcting factor for the current representative point is equal to $K=150.42$ and representative point coordinates are the following:
2. Convergence of $\langle \sigma_{11} \rangle$ depending of the number of periodical cells levels (n) is presented in Figure 10.

Figure 10. Convergence of $\langle \sigma_{11} \rangle$

Take 3 periodical cells layers for following analysis.

**Creation of effective failure surface**

With use of ANSYS APDL macro was written that enables one to create representative points in $\langle \sigma_{11} \rangle<\sigma_{22}^*<\sigma_{12}^*$ 3D space (plane stress) by means of the describe above method. General view of representative points in shown in Figure 11.1 and their projection on $\langle \sigma_{11} \rangle<\sigma_{22}^*$ plane is shown in Figure 11.2.

Figure 11.1. General view or representative points
Algorithm of composite materials analysis includes shift from heterogeneous model to homogeneous. Tensorial-polynomial criterion for homogeneous materials call “effective” tensorial-polynomial criterion. It has the following form:

\[ F^*_{ij} < \sigma_i > + F^*_{ijkl} < \sigma_{ji} > < \sigma_{lk} > + F^*_{ijklmn} < \sigma_{ji} > < \sigma_{lk} > < \sigma_{mn} > + \]

\[ + F^*_{ijklmnp} < \sigma_{ji} > < \sigma_{lk} > < \sigma_{mn} > < \sigma_{pq} > + \ldots = 1 \]  \hspace{1cm} (12)

where \( i, j, k, l, n, m, p, s = 1,2 \).

The geometrical interpretation of this criterion is a closed surface in \( \sigma^\ast_{11}, \sigma^\ast_{22}, \sigma^\ast_{12} \) space. Creation of this surface was done by method of least squares. At this not all points turn out to be located inside this surface. To estimate better order of approximation it is suggested to introduce the value of so-called Average Relative Deviation (ARD) \( \Theta \) of all points of the created surface.

The equation of the sought surface has the following form:

\[ \Omega_{222} = F^*_{ij} < \sigma_i > + F^*_{ijkl} < \sigma_{ji} > < \sigma_{lk} > + F^*_{ijklmn} < \sigma_{ji} > < \sigma_{lk} > < \sigma_{mn} > + \]

\[ + F^*_{ijklmnp} < \sigma_{ji} > < \sigma_{lk} > < \sigma_{mn} > < \sigma_{pq} > - 1 \]  \hspace{1cm} (13)

where \( i, j, k, l, n, m, p, s = 1,2 \).

Relative deviation of the representative point (\( \chi \)) relative to the surface can be evaluated by substitution of representative point coordinates into equation of surface. Sign of \( \chi \) will be negative in case of point hit inside the surface. The closer point will be to the surface, the less \( \chi \) value will be (Figure 12). \( \Theta \) is arithmetical average of \( \chi \):

\[ \Theta = \frac{1}{N} \sum_{i=1}^{N} \chi(i) \]  \hspace{1cm} (14)

where \( N \) - number of representative points (222 in this case).
Figure 12. Definition of x Sign

Effective surface of the second order and its cross-sections projections on $\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$ plane are shown in Figure 13, where number 1 mark cross-sections projection at $\sqrt{\sigma_{12}} = 0$ MPa, 2 – at $\sqrt{\sigma_{12}} = 10$ MPa, 3 – at $\sqrt{\sigma_{12}} = 20$ MPa, 4 – at $\sqrt{\sigma_{12}} = 30$ MPa, 5 – at $\sqrt{\sigma_{12}} = 40$ MPa. For the current surface ARD = -0.0295; 132 points are located inside the surface and 90 points - outside.

Figure 13. Effective surface of the second order

Account of third order items in the equation of failure surface doesn’t make significant changes. Effective failure surface of the fourth order and its cross-sections projections (at $\sqrt{\sigma_{11}} = 0, 10, 20, 30$ MPa) on $\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$ are shown in Figure 14, where number 1 mark cross-sections projection at $\sqrt{\sigma_{12}} = 0$ MPa, 2 – at $\sqrt{\sigma_{12}} = 10$ MPa, 3 – at $\sqrt{\sigma_{12}} = 20$ MPa, 4 – at $\sqrt{\sigma_{12}} = 30$ MPa. For the current surface ARD = -0.0117, 149 points are located inside the surface and 73 points - outside.

Figure 14. Effective surface of the fourth order
Average Relative Deviation for effective failure surface of the fourth order is smaller, then one for second order. As it can be seen, fourth order surface is better fitting all representative points.

Account of the fifth order items in the equation of failure surface doesn’t make significant changes. As the result the fourth order surface describes better the created representative points. Fourth order of the surface is equivalent to the keeping of failure surface tensor up to 8-th rank in the tensorial-polynomial criterion.

At construction of effective surface with use of least squares method some representative points are located inside failure surface. When using this criterion it can lead to the fact that not all critical zones will be determined. To eliminate this weak point the surface was modified by introducing correction reduction coefficient \( k_{ij} \), where \( i \) – the number of representative points taken to construct the surface. In this case \( i = 222 \):

\[
\Psi_{(222)}^{(8)} = F_{y} < \sigma_{y} > + F_{ijkl} < \sigma_{ij} > < \sigma_{ik} > + F_{ijklmn} < \sigma_{ij} > < \sigma_{ik} > < \sigma_{nk} > + F_{ijklmnop} < \sigma_{ij} > < \sigma_{ik} > < \sigma_{nk} > < \sigma_{mp} > + 1 + k_{(222)}
\]

(15)

To calculate effective reduction coefficient it is necessary to define internal point that is located farthest from the surface:

\[
\inf( \Psi_{(222)}^{(8)} - 1)
\]

(16)

where infimum is calculated through all representative points.

Effective reduction coefficient can be found in the following way:

\[
k_{(222)} = 1 - \Psi_{(222)}^{(8)}
\]

(17)

where \( \Psi_{(222)}^{(8)} \) - function of \( \Psi_{(222)}^{(8)} \) with coordinates of the internal point located farthest from the surface.

The obtained in this way coefficient \( k_{(222)} = 0.12 \). Correction of the failure surface is shown graphically in Figure 15. Initial surface (45 points inside) is shown with green, and modified surface (0 points inside) – with blue.

Figure 15. Correction of the failure surface

The obtained effective failure surface fits only for the internal periodical cells. Quite often critical zones in composite structure are located near the boundary. Thus the necessity is arising of the creation of failure surface for the boundary cells. At this it is necessary to consider two types of boundary conditions: kinematical and hybrid boundary conditions, what will give two failure surfaces. First of all consider kinematical boundary conditions. Statement of the problem and FE models are shown in Figure 16. Periodical cells where stress averaging is done are marked with red. In our case periodical cell has rectangular shape, so there are only two available borders types.
Figure 16. Boundary conditions and FE models for two cell locations

Failure surface for boundary cells at free boundary reduces to two representative points on corresponding axes: A and B for the periodical cells near free horizontal border (Figure 17.1), C and D – for the periodical cells near free vertical border (Figure 17.2). These points lay outside the corrected failure surface for inner cells. So it is possible to use this surface for both (inner and boundary) cells types.

Figure 17.1. Representative points for periodical cell, located near horizontal border
Determination of critical zones in composite structures

To illustrate the suggested method unidirectional fiber composite was considered. Material of the matrix – brittle isotropic polymer, fiber – steel. Periodical cell of the composite is shown in Figure 18, where \(a = 25 \, \mu m\), \(b = 20 \, \mu m\), \(c = 15 \, \mu m\). Volume concentration of the fiber \(V_f = 0.3534\). Material properties are listed in Table 3.

![Periodical cell](image)

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<th>(\nu)</th>
<th>(\sigma^0) (MPa)</th>
<th>(\sigma^*) (MPa)</th>
<th>(\sigma_c) (MPa)</th>
<th>(\sigma_T) (MPa)</th>
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In Table 4 effective characteristics obtained with use of direct homogenization method are presented:

![Effective characteristics](image)

<table>
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</tr>
</tbody>
</table>
To analyze working capacity of the developed criterion the reference problem was solved. I.e. problem, where FE model of the composite with account of all microstructure is represented. Statement of the problem and its solution are shown in Figure 19. Comparison of solutions is done is the marked point A.

**Figure 19. Statement of the problem and its solution**

To determine criterion work efficiency it is suggested to introduce co-called 15% range, which includes stresses forming 85% of ultimate stresses. Critical zones where ultimate stress is exceeded defined for micro heterogeneous media (reference solution) and 15% range are presented in Figures 20 and 21.

**Figure 20. Critical zones of reference solution**

**Figure 21. Critical zones of 15% range in reference solution**

Finite element effective tensorial-polynomial criterion (FE ETPC) has the following form:

\[
\Psi^{(8)}_{(222)} = F_{ij} < \sigma_{ij} > + F_{ijkl} < \sigma_{ji} > < \sigma_{lk} > + F_{ijklmn} < \sigma_{ji} > < \sigma_{lk} > < \sigma_{nm} > + \\
+ F_{ijklmnp} < \sigma_{ji} > < \sigma_{lk} > < \sigma_{nm} > < \sigma_{np} > - k_{(222)}
\]

(18)

Critical zones obtained with use of FE ETPC with reference problem FE mesh are shown in Figure 22.1, and with arbitrary FE mesh in Figure 22.2. It should be noted that application of this criterion in some case gives more critical zones then in reference problem. Stresses in the lower zone in Figures 22.1 and 22.2 are close to critical but doesn’t exceed limits. Criterion satisfaction in this zone can be explained by failure surface correction with effective
reduction coefficient. Thus FE ETPC determines under-critical stress zones. Differences between under-critical zones and critical zones depends on effective reduction coefficient $k_{(222)}$.

Figure 22.1. Critical zones obtained with use of FE ETPC

Figure 22.2. Critical zones obtained with use of FE ETPC with arbitrary FE mesh

For more accurate solution definition in the zones of criterion satisfaction it is suggested to use sequential heterogenization, i.e. entering the heterogeneous periodical cell layers in the homogeneous model. The cell, which were touched upon the zones of criterion satisfaction, are taken as 0 layer. Other cell are situated by the concentric layers around 0 layer. Sequential heterogenization and convergence of principal stress in the marked point A depending on the number of heterogenization layers (n) are presented in Figures 23 and 24.

Figure 23. Layers of sequential heterogenization
Figure 24. Convergence of principal stress depending on the number of heterogenization layers

The developed criterion must be satisfied in any case of loading. To verify this it was suggested to solve a number of case studies with various boundary conditions (shear, compression, tension) and analyze working capacity of the criterion in every case.

Case study 1

Let us consider the micro heterogeneous media, presented in Figure 25. Specific character of this task is represented by central hole, which is the stress concentrator. Critical zones of ultimate stress exceeding defined for micro heterogeneous media (reference solution) and 10% range (range includes stresses forming 90% of ultimate stress) are presented in Figure 26.

Critical zones obtained with use of FE ETPC with arbitrary problem FE mesh are shown in Figure 27.

Figure 25. Statement of the problem 1

Figure 26. Critical zones of reference solution for problem 1

Critical zones obtained with use of FE ETPC with arbitrary problem FE mesh are shown in Figure 27.
Case study 2

Then it is suggested to consider the micro heterogeneous media with non symmetric load, which is presented in Figure 28. Critical zones of ultimate stress exceeding defined for micro heterogeneous media (reference solution) and 10% range (range includes stresses forming 90% of ultimate stress) are presented in Figure 29.

Critical zones obtained with use of FE ETPC with arbitrary problem FE mesh are shown in Figure 30.
Case study 3

Finite element statement of the third case study is presented in Figure 31. Critical zones of ultimate stress exceeding defined for micro heterogeneous media (reference solution) and 10% range (range includes stresses forming 90% of ultimate stress) are presented in Figure 32.

Critical zones obtained with use of FE ETPC with arbitrary problem FE mesh are shown in Figure 33.
Conclusions

The new method of critical zones estimation in composite materials was suggested. Main steps of this method are:

1. Homogenization of composite material – computation of effective mechanical characteristics and failure surface tensors;
2. Representative points were constructed to build failure surface. Ground for failure surface order choosing and necessity of its modification are presented.
3. Macro analysis in the suggested construction of determination of critical zones;
4. Sequential heterogenization in critical zones and evaluation of micro stresses.

The developed method was applied to the composite structure (plane stress) and detailed development of the method algorithm was done. The method was also verified in case studies of complex loading of composite structure. As further development of the methods it can be expanded to the analysis of 3D constructions.

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